The Guideway and Posts
of the
Intelligent Transportation
Network System
ITNS

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Contents

How to Design a PRT Guideway

Abstract

The guideway is the most expensive item in a PRT system. Yet in all but a few cases the design of the guideway was more or less an afterthought – something that did not require a great deal of attention. This is a major reason many PRT systems have not survived. Primary attention had to be placed on the development and design of the control system because it was the single technological advance that made consideration of PRT possible. With limited resources, control downgraded the importance of everything else about a PRT system. During the long history of PRT development and design, guideways have been designed for Veyar, Monocab, TTI, StaRRcar, Uniflo, Dashaveyor, Morgantown, The Aerospace Corporation PRT System, Cabintaxi, CVS, Aramis, ELAN-SIG, VEC, Swede Track, Mitchell, SkyCab, Taxi 2000, PRT 2000, Microrail, Skytran, MonicPRT, ULTra, Vectus, and others. This plethora of designs likely has had much to do with the reluctance of city planners to recommend PRT. No two of these guideway designs are very close to each other. Now that the control problem is well understood, it is time to turn more attention to the guideway. The purpose of this paper is to stress the importance of adequate consideration of guideway design requirements and criteria as the basis for the design of guideways that have the potential of becoming standardized and widely deployed.

Introduction

As an engineering professor working on PRT for 13 years with no commitment to any particular system, I was privileged to visit the inventors and developers of Veyar, Monocab, TTI, StaRRcar, Uniflo, Dashaveyor, Morgantown, The Aerospace Corporation PRT System, Cabintaxi, CVS, Aramis, ELAN-SIG, VEC, Swede Track, Mitchell as well as other AGT systems then under development including Westinghouse Skybus, Jetrail, Airtrans, Ford-ACT, UTDC, Universal Mobility, H-Bahn, Krauss-Maffei, VAL, and AGRT. Later I developed Taxi 2000 and watched in dismay as it degraded into PRT 2000, mainly because guideway design was not taken seriously. Later I learned of Austran, Cybertran, SkyCab, Microrail, Skytran, MonicPRT, ULTra, and Vectus. Now there are many more offerings than I can name. Some of these systems were on paper only, some were built as test tracks, and some were built as applications, but they all provided opportunities to become aware of the variety of guideway designs.

At the University of Minnesota early in my work on PRT I coordinated a Task Force on New Concepts in Urban Transportation. We conducted planning studies of PRT for Minneapolis, St. Paul, and Duluth and soon saw that such studies were mandatory to real understanding of the problems of designing and installing a PRT system, including its guideway. We discussed our work with many public officials, planners, and interested citizens not only in Minnesota, but in many locations around the United States, Canada, Europe, and Asia. We reviewed the work of the many government-funded studies related to AGT design. The most helpful for guideway design were [Snyder, 1975], [Stevens, 1979], and [Murtoh, 1984]. Out of this experience, I was able to write down a hopefully comprehensive set of requirements and criteria for the design of a PRT guideway, and subsequently found a design configuration that met them all. The discussion in this paper applies to <u>elevated</u> guideway structures for the simple reason that after trading off underground, surface-level, and elevated systems planners almost always opt for elevated systems.

As overall guidance for guideway design I find it difficult to improve on the following statement [Pushkarev, 1982] by Louis J. Gambaccini, New Jersey Transportation Commissioner and creator of the nation's first statewide public transit agency.

> "Fixed guideway transit is not a universal solution nor should it be applied in all urban areas. Fixed guideway is a potential strategy, as is the bus, the ferry boat, the car pool or the van pool. In many possible applications, fixed guideway is a superior strategy. But whatever strategy is finally selected, each should be evaluated not in the narrow context of transportation alone, nor solely in the framework of accounting. It should be measured in the broader context of its contribution to the overall long-term aspirations of the urban society it is supposed to serve."

Our challenge today is to design and build PRT systems even more able to "contribute to the overall long-term aspirations of the urban society" than Mr. Gambaccini could imagine thirty years ago.

Definitions

From the Oxford American Dictionary:

A Need: A circumstance in which a thing or a course of action is required.

A Criterion: A standard of judgment.

An Attribute: A quality that is characteristic of a person or thing.

From Wikipedia:

A Requirement: A necessary attribute, capability, characteristic, or quality of a system in order for it to have value and utility to a user.

Design Process

After decades of experience in the practice and teaching of engineering design I realized that the first step in a design process is to study deeply and follow rigorously a comprehensive set of rules of engineering design. I make no claim that my set [Anderson, 2007a] of such rules is complete, and I welcome collaboration with other experienced engineering designers to develop a more comprehensive set. But I have observed that the less successful PRT guideway designs have resulted primarily from violating one or more of these rules. What is now commonly called "risk management" consists mainly in following rigorously such a set of rules. My contribution was inspired by reading, as a young design engineer, the Rules of Engineering of W. J. King [King, 1944]. Beginning with these rules, the design processes I used to arrive at my conclusions about the design of a PRT system are summarized in a DVD [Anderson, 2008d].

The next step is to write down a simple statement answering this question: What does a PRT guideway really need to do if it is to win competitions? Here is my short answer:

> A PRT guideway must carry vehicles containing people safely, reliably, and comfortably in all reasonable environmental conditions for up to 50 years over curves, hills, and straight sections at an acceptable range of speeds, acceptable cost, and acceptable visual impact.

But, we need to be more specific. Only by long experience in the design of whole PRT systems can one unearth all of the requirements and criteria for guideway design. Designing a PRT guideway cannot be done successfully without a great detail of development work on the whole system because the guideway design depends on other system features and other system features depend on guideway specifics [Anderson, 2000, 2008a]. In the following section, in no particular order, I give my list of guideway design requirements. All are important. To be successful, none can be ignored. For clarity and ease of reading, I list the requirements for the design of an elevated guideway without comment and without quantification. I then discuss alternative system issues and tradeoffs that in some cases affect guideway design and in others are influenced by the guideway-design requirements. Next, I list three guideway-design tradeoffs. Then, I suggest design criteria. Finally, I state how, by using this process, I arrived at my guideway design. My bottom line goal for decades has been to design a system of urban transportation that can recover all of its costs from revenue – to turn urban transportation into a profitable enterprise.

PRT Guideway Design Requirements

1. The guideway must assure an acceptably high level of safety for the passengers that ride in the vehicles mounted on it in all reasonable circumstances.
2. Consistent with other requirements, the guideway must have minimum size, weight and capital cost.
3. The appearance of the guideway must be acceptable and variable to suit the community.
4. The switching concept for merge and diverge sections of the guideway must be straightforward, easily explained, and one of the first items to clarify while developing the configuration.
5. Accommodation of hills, valleys, and horizontal curves must be straightforward.
6. The design must permit straightforward manufacturability and installation.
7. Ride comfort must be acceptable.
8. The design must be compatible with the Americans with Disabilities Act.
9. The guideway must be designed to minimize operating cost.
10. The minimum span length must be determined from careful city planning.
11. The guideway must be designed for long life under the variable vertical, lateral, and longitudinal loads that can reasonably be expected.
12. The guideway must be designed to withstand reasonable earthquake loads.
13. There can be no passenger injury due to collisions of street vehicles with support posts, falling trees, etc. if such events may be possible.

14. The system must be designed to operate in the presence of wind, rain, snow, ice, lightning, dust, salt and other airborne corrosive substances, nesting birds and insects, i.e. in a general outdoor environment.
15. The guideway must be designed so that under winter conditions, guideway heating will not be necessary, except for systems not intended to operate under winter conditions.
16. The guideway must be easy to erect, change, expand, or remove.
17. The guideway design must permit access for maintenance.
18. The guideway must be designed for relief of thermal stresses.
19. The guideway must be designed for competitive operating speeds.
20. The guideway design must permit the system to expand indefinitely.

21. If power rails are used, the guideway must be designed so that frost will not form on them.
22. It must be very difficult if not impossible for anyone to be electrocuted by the system.
23. The guideway must be designed with adequate torsional stiffness.
24. It must be very difficult if not impossible to walk on the guideway.
25. The guideway design must liberalize the required post-settling tolerance.
26. The guideway design must eliminate slope discontinuities.
27. There must be space in the guideway for the communication means.
28. The design must minimize electromagnetic interference.
29. The design must minimize acoustical noise.
30. The design must minimize the potential for vandalism or sabotage
31. Provision must be made in the guideway design to prevent corrosion.
32. There must be no place in the guideway for water accumulation.
33. The design must provide for vibration damping.

Issues and Tradeoffs in PRT System Design

Early in my career at the University of Minnesota, I was privileged to hear a lecture by California Institute of Technology Professor Fritz Zwicky, in which he stressed "the morphological approach which attempts to view all problems in their totality and <u>without prejudice</u>." During World War II, he was deeply engaged in the design of jet engines, in which process, before any detailed design was begun, he and his colleagues wrote down in chart form every way they could conceive that a jet engine could be designed. The process described in his book [Zwicky, 1962] is general. It is a useful guide to the design of anything, and it strongly influenced the way I taught engineering design and in the methodology I practiced in the design of my PRT system. Zwicky's influence is present in the preceding and following discussion. One makes progress by "standing on the shoulders of giants." Zwicky was one of the giants. Here are some of the results of morphological thinking:

Safety issues. These issues are mentioned because they need to be treated as part of the overall PRT system design process. Neglecting any one of them can result in rejection. Discussion of the details is, however, beyond the scope of this paper. [Irving, 1978; Anderson, 1978a; Anderson, 1994]

a. How can the control system be designed for maximum practical safety?
b. How can the vehicles be designed for maximum practical safety?
c. What should be the minimum operational headway?
d. Should seat belts, air bags, or neither be required?
e. Should shock-absorbing bumpers be designed into the vehicles?
f. How should potential collisions with street vehicles or other objects be handled?
g. How can people be prevented from walking on the guideway?
h. How can the possibility of electrocution be prevented?
i. How should fire safety issues be handled? NFPA 130.
j. How should evacuation and rescue be handled?

2. Is the system predominately elevated, at grade, or underground? The issues are

    a. Congestion relief
    b. Safety
    c. Land requirements
    d. Costs

3. Is a walkway along the guideway necessary?

    This issue has been debated for a long time [NFPA 150, Anderson, 1978b]. If one or more vehicles are stranded on the guideway, how should passengers be rescued? The requirement of a walkway will make the guideway larger and more expensive, for which reason the guideway designer would like not to be required to include walkways. There are two essential subsidiary considerations that must be understood:

    a. Can all kinds of people including the elderly and the disabled in all reasonable kinds of weather use a walkway? Could a walkway be acceptable in rainy, snowy, or windy conditions? A little reflection shows that a walkway would be usable for the abler bodied people in a warm and dry climate, and thus, if PRT is to be acceptable for all people, it must be possible to design the system in such a way that the mean time between incidents in which a walkway would be desirable is long enough to be acceptable [Anderson, 2006], and in the remote situation in which someone might need to be rescued a means other than a walkway is acceptable.

    b. Can the system be designed in such a way that the mean time between circumstances in which a walkway would be useful is so rare that other rescue means become acceptable?

    These questions were studied in sufficient detail in the Chicago PRT Design Study[1] that it was concluded that walkways would not be required except in circumstances such as river crossings. When there is ground underneath the guideway, the preferred alternative rescue means would be a fire truck or a cherry picker. Even when crossing rivers, detailed work on analysis of hazards and potential failures and their effects [Stone & Webster, 1991] resulted in the conclusion that rescue could best be accomplished by means other than a walkway. The study team concluded that PRT systems can be designed to be sufficiently simple and reliable that walkways will not be needed.

4. Should the system be dual mode or single mode, i.e., with vehicles captive to the guideway? This question has been studied [Irving, 1978; Anderson, 2007b] in sufficient detail to convince us that we should concentrate on single-mode PRT systems. We considered many issues including

[1] Formally, the Northeastern Illinois Regional Transportation Authority PRT Design Study of 1990.

    a. The effect on community development patterns.
    b. The effect on system cost and ridership.
    c. The effect on capacity.
    d. The effect on those who cannot, should not, or prefer not to drive.

5. Should the vehicles be supported above the guideway or should they hang below? This is a complex tradeoff that I have examined in increasing detail [Anderson, 2008b]. The issues are:
    a. Visual impact
    b. System cost
    c. Natural frequency
    d. Ease of switching
    e. Rider security
    f. All-weather operation
    g. Torsion in curves
    h. Motion sickness

6. How should the vehicles be suspended? [Anderson, 2008c]

    a. Wheels
    b. Air cushions
    c. Magnetic fields

7. How should the vehicles be propelled? [Anderson, 1994; 2008d]

    a. Rotary motors
    b. Linear motors
        i. Induction
        ii. Synchronous
        iii. Air
        iv. Rope

8. What should be the people-carrying capacity of the vehicles? [Anderson, 1986]

    a. Understand the size of groups in which people travel.
    b. Understand the ease of taking two or more vehicles.
    c. Understand the effect of vehicle size on system cost.
    d. Need to accommodate wheelchair + attendant, bicycle, baby stroller, <u>or</u> luggage.

9. Assuming electric motors, should they be rotary or linear? [Anderson, 1994]

10. Should the motors be on board the vehicles or at wayside? [Anderson, 2008d]

11. If the motors are on board, should they draw power from batteries or power rails? [Anderson, 2008d]

All of these tradeoffs and more will affect the cost and performance of the system and should be studied very carefully before detailed design is initiated.

Tradeoffs in PRT Guideway Design

1. Cross sectional dimensions: The minimum-weight cross section should be used. [Anderson, 1978, Chapter 10; 1997; 2007c]
2. Material: Steel, concrete, composite?
3. Truss or plate or pipe?

PRT Guideway Design Criteria

1. Vertical and Lateral Design Loads. This is the only set of criteria considered by Moutoh, 1984. One must consider dynamic loading due to vehicles moving at speed, wind loads, earthquake loads, longitudinal loads due to braking vehicles, and loads due to street vehicles crashing into the support posts, if that is to be permitted. The best study I have seen on dynamic loads is one done in the M. I. T. Mechanical Engineering Department by Snyder, Wormley, and Richardson [Snyder, 1975]. In their computer studies, they simulated vehicles of various weights operating at various speeds and various headways, and running over guideways of various span lengths. By placing their results in dimensionless form, the usefulness was extended considerably. I studied their results [Anderson, 1978a] and noted that the shorter the minimum headway the smaller was the difference between dynamic and static deflection, and in the theoretical limit of zero spacing between vehicles the dynamic and static deflection are the same, i.e., the guideway cannot tell the difference. Assuming PRT vehicles operating at a minimum headway of half a second, I found that the maximum dynamic guideway deflection and stress with vehicles operating at line speed was less than the maximum deflection and stress with vehicles nose-to-tail on the guideway. Therefore, the maximum possible vertical load becomes a uniform load and it is easiest to calculate. The loading criteria used in the Chicago PRT design study were

   1) Fully loaded vehicles nose to tail on span + 30 m/s (70 mph) crosswind.
   2) No vehicles + 54 m/s (120 mph) crosswind. [I now assume 80 m/s (180 mph)]

   The maximum wind load on a guideway can be substantially reduced by reducing its drag coefficient based on known wind-tunnel data [Hoerner, 1965], [Scraton, 1971].

2. Longitudinal loads. The criterion is based on vehicles operating at minimum headway all stopping simultaneously at 0.5 g. I found this load to be less than the maximum wind load.

3. Earthquake load. There is debate on the maximum horizontal acceleration measured due to an earthquake. In a presentation at a Society of American Military Engineers conference in San Diego in the last week of March, 1994, shortly after the Los Angeles earthquake, an

Army Major General who had been placed in charge of rebuilding the Los Angeles freeways told his audience that the maximum horizontal acceleration measured was 1.6g, which is higher than any figure I have seen in print. The bottom line, though, is that the lighter the elevated structure, the easier it is to design foundations to withstand such loads. I have found that for the guideway I designed a horizontal acceleration of the ground of 0.86 g is equivalent to a wind load of 80 m/s (180 mph). A PRT guideway must be designed to the local earthquake code, which varies considerably from one region to another.

4. Design stress – The designer must use standard values for the selected material.

    a. Specify corrosion protection for the life of the structure.
    b. Prevent water accumulation.
    c. Plan to clean out any bird droppings, which are corrosive.
    d. Design to account for material fatigue over the specified life.
    e. Design to relieve thermal stresses.

5. Maximum allowable deflection. The AASHTO bridge standard is span/800.

6. Minimum allowable span. The Chicago PRT design study conclusion: 28 m (90 ft)

7. Ride Comfort

    a. Observe the ISO standards for acceleration vs. frequency
    b. Observe the ISO standard acceptable constant acceleration and jerk for normal and emergency operation, which are also given in the ASCE APM Standards.

8. System Life. The Chicago RTA specified 50 years.

9. Compliance with the Americans with Disabilities Act (ADA).

    a. Must accommodate a wheelchair with an attendant.
    b. In the Chicago study, the disability community strongly demanded access to every vehicle, with the wheelchair facing forward.
    c. Must provide for visual and hearing disabilities.

10. The minimum line headway needs to be specified at the beginning of the design program based on detailed site-specific planning studies. When it is not, as has usually been the case, the system may be destined for a limited range of applications. Based on many independent studies we have designed for a minimum headway of half a second. [Anderson, 1994]

11. Design for the expected environment

    a. Rain, ice, snow of a given rate of accumulation.
    b. Ambient temperature range, typically -40^{o}C to +50^{o}C.
    c. Lightning protection.

d. Sun.
e. Dust, sand, salt.
f. Nesting bees, birds, squirrels, etc.
g. Earthquakes – Design to maximum expected horizontal acceleration at the site.
h. Fire. [NFPA 130]
i. Vehicles crashing into posts. [Anderson, 2006, Appendix A]
j. Interference from other elements of the urban scene.
k. Ice build-up on power rails due to clear winter night sky.

12. Speed range. Select the cruising speed to minimize cost per passenger per unit of distance. Consider that turn radii, stopping distance, kinetic energy, and the energy needed to overcome air drag all increase as the square of speed; and that energy use depends on streamlining, low road resistance, and propulsion efficiency. Consider that the maximum operational speed for acceptable ride comfort is proportional to the guideway natural frequency, which depends on guideway stiffness and the type of support. [Anderson, 1997]

13. Costs. The design team should aim for costs sufficiently low to be recoverable in fares, i.e., the system should be designed to be a profitable private enterprise. Such a conclusion clearly cannot be reached without a great deal of development work, but by striving for this goal the design team will insure its future.

14. Require a small amount of vibration damping in the guideway.

15. Acoustical noise should be less than the noise of automobiles on streets.

16. Electromagnetic noise generated cannot interfere with existing devices.

17. Communication means must be accommodated.

18. Expansion. Design so that the system can be expanded indefinitely.

19. Design to minimize the effects of vandalism and sabotage.

    a. Assign young engineers to study ways to vandalize the system and how to prevent it.
    b. The spread-out nature of a PRT system provides no inviting target.

My Conclusions [Anderson, 2007c, 2008a, 2008d]

1. Resolving the basic tradeoffs related to the guideway, I reached the following conclusions:

    a. The guideway will be mostly elevated.
    b. Single mode.
    c. Supported vehicles.
    d. Wheeled suspension.

e. Linear-induction-motor propulsion.
f. Motors on board, powered via power rails.

2. Before designing the guideway, determine the vehicle maximum weight with careful weight-minimization design.

3. Use the optimum guideway cross section for minimum weight and cross sectional area.

    a. The optimum guideway is narrower than it is deep.
    b. A vertical chassis is required.
    c. Careful attention must be given to the attachment of the cabin to the chassis. Detailed finite-element analysis gives a practical solution.

4. The minimum-weight, minimum-size guideway is a steel truss.

    a. Robotic welding is required for acceptable cost.
    b. Corrosion protection is required.
    c. The guideway should be clamped to the posts for maximum stiffness.
    d. Expansion joints should be placed at the point of zero bending moment in uniformly loaded spans.

5. Cover the truss with composite covers, opened 3 in at top, 6 in at bottom, with curve radii at top and bottom 1/6th guideway height, hinged at bottom and latched at top, with a thin aluminum layer and sound-deadening material on the inside [Anderson, 2008a]. The benefits are:

    a. The interior of guideway is protected from all but very minimum snow and ice.
    b. The interior is protected from effects of the sun on the tires and other equipment.
    c. Differential thermal expansion is eliminated.
    d. The exterior environment is shielded from electromagnetic and acoustic noise.
    e. The power rails are protected from the winter night sky, which prevents ice accumulation.
    f. Wind drag is 30 % of that on an opened truss. [Scraton, 1971]
    g. The interior of the guideway can be accessed for maintenance.
    h. The appearance of the guideway can be selected to suit the community.

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The ITNS Guideway and its Interface with the Vehicle Chassis

Contents

1. <u>The Guideway Cross section</u>

The guideway design requirements and criteria upon which the design of this guideway is based are given in the paper "How to Design a PRT Guideway," which was presented at the Automated People Mover Conference in Atlanta, Georgia, in June 2010. Detailed calculations that define the guideway are given in the document "Structural Properties of the ITNS Guideway." Figure 1 is a drawing of the guideway cross section and the vehicle's two-inch-wide vertical chassis. Figure 2 shows drawings of the three orthogonal views of the guideway. Figure 2 shows front and side views of the vehicle attached to the guideway.

2. <u>The U-Frame and Stringers</u>

The basic element of the guideway structure is an assembly consisting of vertically oriented 5" × 9 lb/ft channel sections welded diagonally at the two lower corners to form a U-shape. One of these U-shaped frames is placed every 54 inches along the direction of the guideway. With a sufficiently high production quantity, these U-frames could be stamped out of sheet steel. The internal width of the U-frames is 22 1/8" plus a small manufacturing tolerance, and the distance between the top and bottom of the U-frame is 34 1/4". In each of the four corners of the U-frames a cut-out permits installation of a 4-inch OD square or round thin-walled tube, which is welded to the U-frames. In straight sections the wall thickness of these square tubes is 0.174" and in curved sections 0.315", which are standard sizes taken from the *Manual of Steel Con-*

struction. Use of channel sections for the U-frames provides convenient surfaces to which to attach the power rails, the leaky cable, and the assembly of main-wheel running surfaces. This assembly of U-shaped frames and tubes will be called the basic guideway structure (BGS).

3. The Main-Wheel Running Surfaces

Just above the bottom horizontal element of the U-frames, a pair of facing angles 8" wide × 6" high × 1/2" thick form the running surfaces for 13 ¼" OD × 4" wide main vehicle-support tires, the running surfaces for 6" OD × 2" wide lower-lateral polyurethane support tires, and the reaction surface for a pair of linear induction motors (LIMs) that act against the horizontal surfaces. The inner vertical surfaces of this pair of angles, being the running surfaces for the lower set of lateral support tires, must be held by means of gage blocks to 21 inches. The 21-inch dimension is determined by the needs of the pair of LIMs. We will call these the "running-surface angles" (RSA).

4. The Copper Sheet

The LIMs require that a 0.080" thick by10" wide copper sheet be attached to each RSA in such a way that 7" of the 10" width be on the top surface. The copper sheet is wrapped around the inside leg of the angle and back 2.5" under the angle. Since the horizontal leg of the running-surface angles is 8" wide, a 5.5" wide steel-to-steel surface is available for welding.

5. The Running-Surface Assembly

After attaching the copper sheets, the pair of RSA is assembled by welding 2.5" × 2.5" ×1/4" × 22" long transverse angles (TA) underneath and perpendicular to the RSA at the half-way point between the U-shaped members, and thus 54" apart. This assembly of RSAs is then laid into the already-formed guideway structure. The running-surface assembly can be either welded[3] or bolted to the U-frames. If welded, the RSAs can be welded directly to the bottom element of the U-frames along the available 5.5" dimension mentioned above. If bolted, a 2.5" × 2.5" ×1/4" × 22" long angle must be welded to each U-frames in a position so that the vertical surface of the transverse angle is right next to the vertical surface of the horizontal element of the U-frame channel section. (Except for tolerances, these angles could be welded in place before the RSA are attached to the BGS.) These two vertical surfaces are then bolted together.

The angles at the half-way point between the U-frames have two functions: They permit the RSA to be combined into an assembly in such a way that the 21" spacing between the inner surfaces of the angle pair is maintained accurately, and they reduce twisting of the RSA under the load of the main-support tires sufficient to insure adequate ride comfort. This result has been determined by calculation.

[3] Pressure welding should be used to minimize warping.

6. The Lower Side Wheels and their Running Surfaces

The horizontal centerline of the lower side support wheels is 1.5” above the top surface of the horizontal leg of the RSA.

7. The Upper Side Wheels and their Running Surfaces

The upper lateral side wheels run against 4” × 4” × ¼” angles, which are welded to the top of the U-frames as shown in Figure 1. The inner vertical surfaces of the left and right angles must be adjusted by means of gage blocks to the 21” dimension mentioned above and positioned directly over the 21” dimension in the lower lateral running surfaces. The horizontal centerline of the upper side support wheels is 2” below the top of the 4” × 4” × ¼” angles. Thus the nominal distance between the upper and lower centerlines of these side-support tires is 24.5.”

8. The Switch

The axis of rotation of the switch arm is 2” above the midpoint of the upper and lower side-wheel centerlines, which places it 12.25” below the top of the U-frame. The switch wheels are 4” OD x 1.5” wide polyurethane. They function against switch rails, which are non-standard 5” wide channels formed from ¼” thick steel plate with the vertical outside dimension 2.25”. The running surface of the switch rails must be smooth and vertical. Because of the close tolerance required between the running surface of the switch rail and the upper and lower lateral running surfaces, it is likely necessary to split the switch rail on its top surface, adjust the position and bolt it in place. The top of the switch rails is 11.5” below the top of the U-frame. The switch is made bi-stable by means of a leaf spring mounted as shown on the drawing in Figure 1. There will be a pair of snubbers positioned to stop the switch arm in its engaged position on either side of the guideway. Figure 1 shows the switch in the engaged position on the right side. There will also be a pair of proximity sensors appropriately positioned to inform the on-board computer of the switch position.

9. Power Rails

Two 600-volt D.C. power rails are mounted in the space above the switch rails so that the following three distances are equal: 1) the distance between the upper side of the switch rail to the lower edge of the lower power rail, 2) the distance between the adjacent edges of the two power rails, and 3) the upper edge of the upper power rail and the lower surface of the upper lateral-wheel support bracket. The minimum separation of these surfaces is a standard specified by electrical manufacturers to avoid arcing. Use of D.C. vs. A.C. power rails permits conversion of utility power to D.C. at wayside, which moves that power conversion equipment from small units in each vehicle to large units at wayside. Providing power from wayside rather than from on-board batteries permits substantially higher operating speeds and permits construction of guideway networks of any extent with no fear of running out of power. In diverge sections of the guideway there must be power rails on both sides.

10. Guideway Covers

Guideway covers are positioned as shown in Figure 1 with curve radii at the four corners of 6”, which is needed to minimize the side air-drag coefficient on the guideway, which minimizes the wind load on the guideway. The left and right surfaces of the covers are shown slightly bowed out or curved to increase stiffness. The lower surfaces are sloped downward to the center to permit water runoff. The covers will be composite with a thin layer of aluminum sprayed on the inside to provide electromagnetic shielding. The covers are hinge attached at the lower edges so that they can be swung downward to permit access to the interior of the guideway in the unlikely circumstance that that would be necessary. The upper surfaces are similarly sloped as shown to assist in runoff of rain or removal of snow. They are attached at the top with suitable clips to keep them firmly in position. The gap at the top is 3” and at the bottom 6”. The outer dimensions of the cover from left to right is 36” and from top to bottom 38”.

11. Communication Cable

Communication between each vehicle computer and a wayside computer is provided by means of a leaky cable. The leaky cable and the bracket that holds it in position is located as shown in Figure 1. In branching sections of the guideway there will be leaky cables on both sides. Cell phone technology could be used for communication if it can be guaranteed that there will be no interference either by hackers or by an electromagnetic pulse, which seems unlikely.

12. Issues

Two issues must be addressed before the guideway structural analyst can proceed to developed final drawings based on computer analysis of the stresses and deflections under agreed loading conditions.

1. Power rails. Electrical standards determine the minimum distance between the power rails and surrounding surfaces. These distances must be approved by a licensed electrical engineer. It is likely that a third grounding rail will be needed. We assume that it could be a brush operating against the inner upper 6” high surface of the main-support angles. It must be approved by a licensed electrical engineer.
2. Leaky cable. We believe the positioning and attachment of the leaky cable will be adequate, but it must be approved by a licensed electronics engineer familiar with the properties of such cables as our primary communication means between the vehicles and wayside computers.

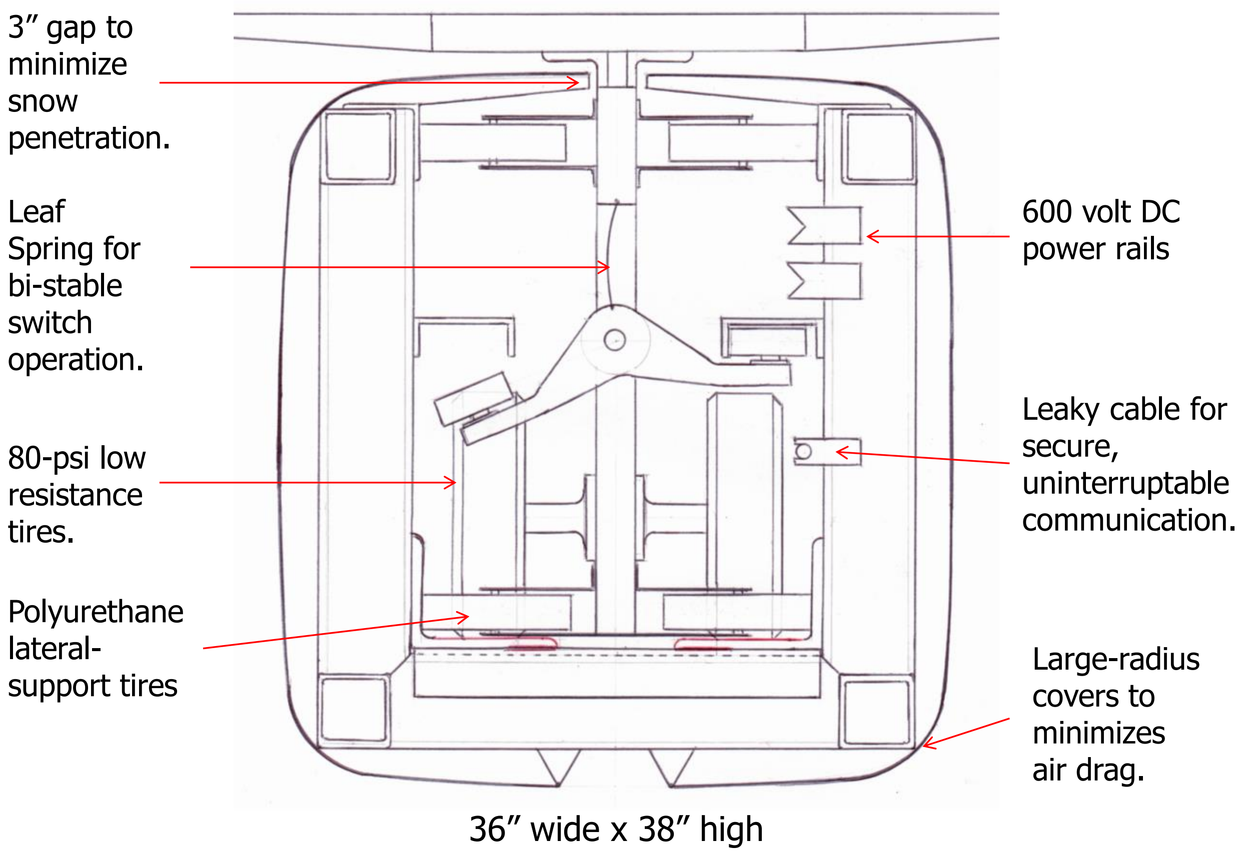

Figure 1. Guideway Cross Section.

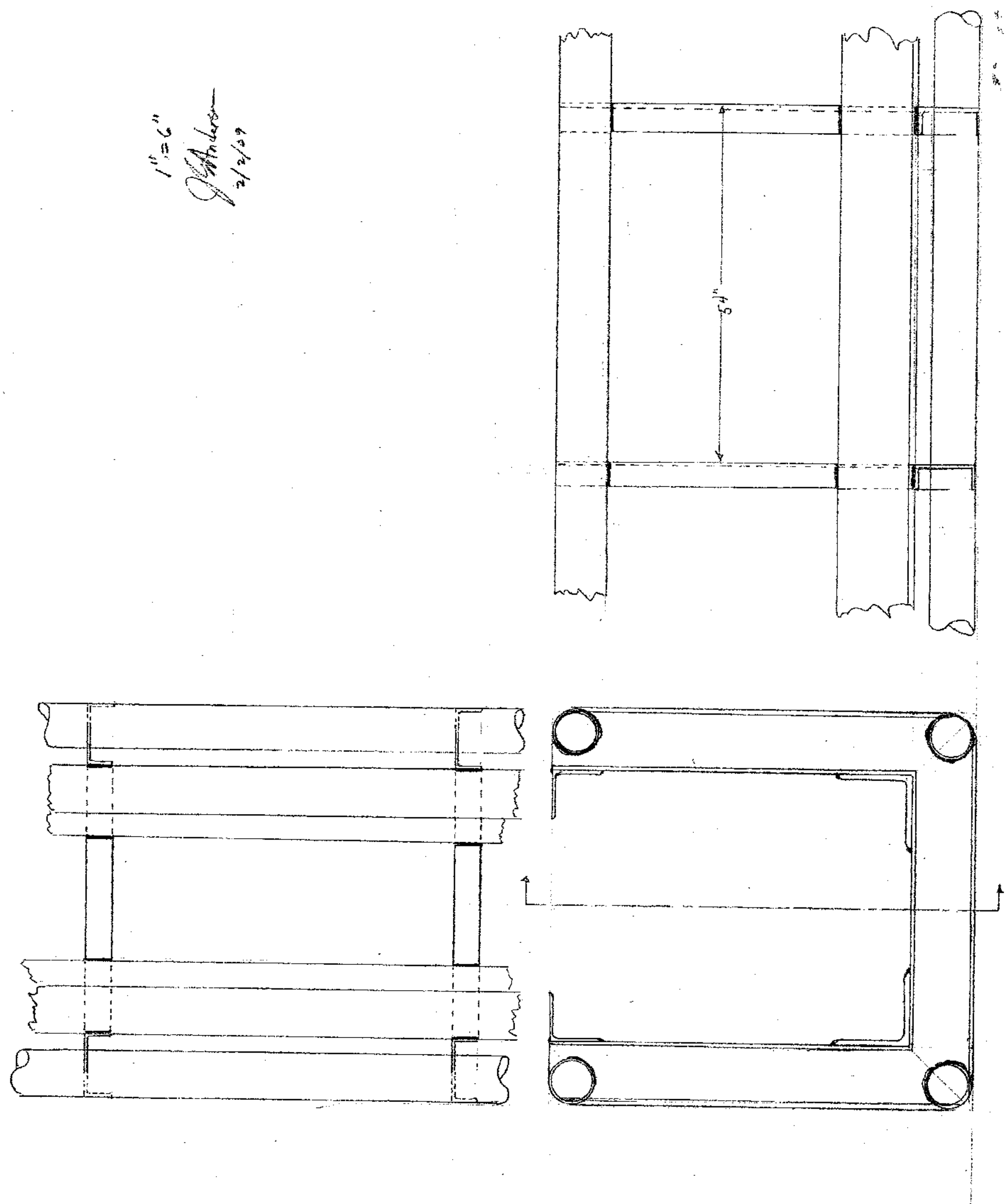

Figure 2. Guideway Elements.

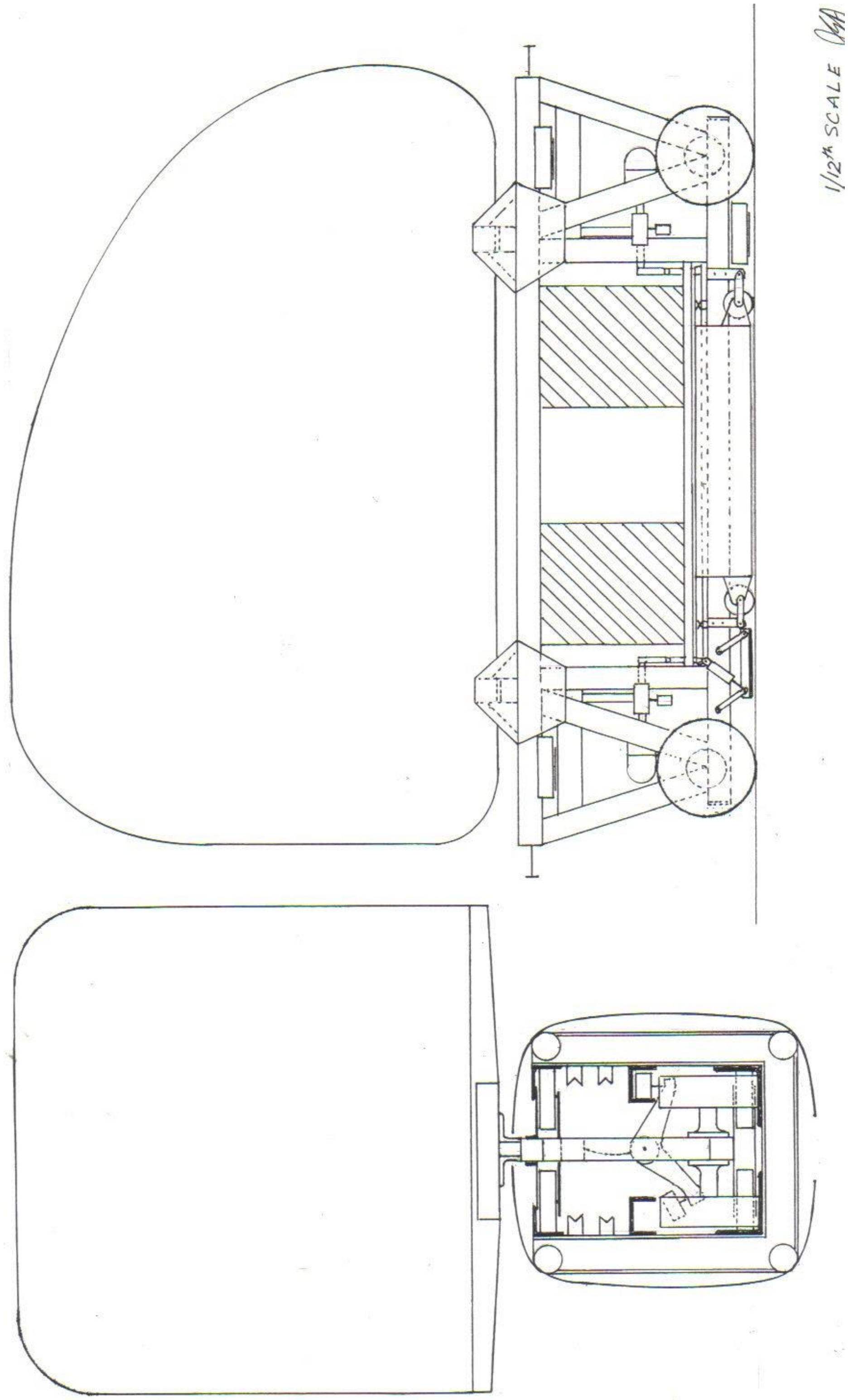

Figure 3. Front and Side Views of the *ITNS* Vehicle.

The Structural Properties of the *ITNS* Guideway

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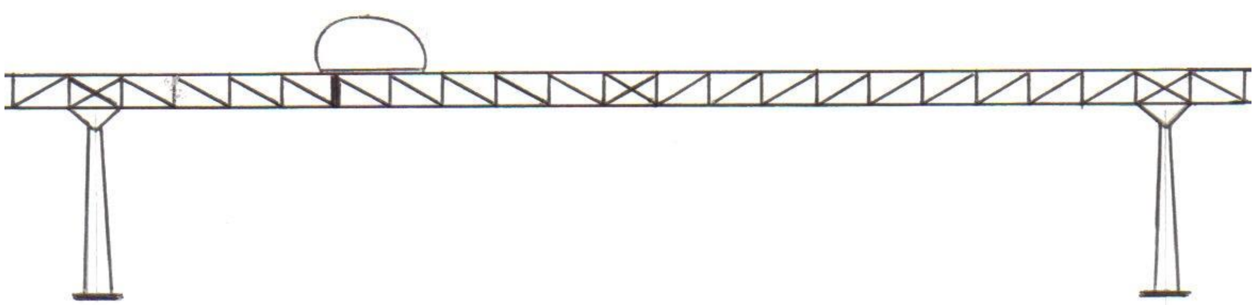


Figure 1. The Truss Guideway

1. Introduction

The purpose of this document is to derive the parameters of the ITNS guideway in the detail that is practical using classical, i.e., non-computerized, methods. The guideway with its covers is designed to meet 37 requirements.[4] Computerized analysis of the structure will likely cause modification of some of the parameters, but it is always desirable to begin with a realistic approximation that can easily be checked. The guideway configuration is shown in Figure 1. It has vertical members every 4.5 feet and is designed for 90-ft spans. It is an evolution of a U-shaped guideway design developed by The Aerospace Corporation for their PRT system[5]. The present design is a truss structure, which reduces the weight per unit length of the guideway by a factor of about three (3) compared to a plate or tube guideway. The running surfaces

[4] J. E. Anderson, "An Intelligent Transportation Network System," April 2015.

[5] Jack H. Irving et al, *Fundamentals of Personal Rapid Transit*, Lexington Books, D. C. Heath and Company, 1978.

and stringers are secured directly to U-frames shown in Figure 2. Curved sections of the guideway will be assembled and welded in a computerized and adjustable jig that will allow any curved section to be built.

The design of the guideway is summarized in Reference 1. The main running surfaces are the pair of 8x6x1/2 inch angles shown in Figure 2. The half-inch dimension of the angles is specified by the needs of the Linear Induction Motors. The main conclusion of Reference 2 is that the design can be based on a static vertical load of loaded vehicles placed nose-to-tail on the guideway because this load surpasses the dynamic load of vehicles operating at minimum headway. Reference 3 shows that ride comfort on a sprung seat is satisfactory if the maximum deflection between supports is less than 0.055 in. Reference 4 shows that the maximum deflection will be about 0.023 in. Reference 5 derives the formula for the midspan deflection. Reference 6 shows that any slope discontinuity between guideway sections must be less than about half a degree. Reference 7 develops details of the joint between guideway sections needed to eliminate slope and step discontinuities. Reference 8 shows that the expansion joint in the guideway must be designed to carry 6% of the maximum bending moment in the guideway. Reference 9 develops a computer program needed to determine maximum wheel loads on the guideway. Reference 10 specifies the switch position at 2 inches above the half-way distance between the upper and lower side wheels and estimates the maximum side-wheel loads. Reference 11 calculates properties of a guideway needed to move vehicles at higher speeds.

2. <u>Description of the Cross Section</u>

The paper "The ITNS Guideway" includes drawings of the guideway and the vertical chassis, to which are attached main-support wheels, lateral stabilization wheels, and switch arms. The major dimensions are shown in Figure 2.

The elements of the guideway are as follows:

1) A series of vertically oriented steel channel-section U-shaped frames spaced 54 inches apart and machined to the inside left/right dimension of 22" +0.010" to - 0 in the areas where upper and lower running surfaces are attached.[6] To assemble the guideway, the U-frames are placed in a computerized fixture in correct positions corresponding to the required guideway curves.

2) Lower running surfaces consist of 8x6x1/2" steel angles oriented with the 8" dimension horizontal. These angles form a) the running surfaces for 13" OD main support tires, b) the running surfaces for 6" OD lower lateral polyurethane tires, and c) the reaction surfaces for the LIMs. The 6" vertical leg of these angles is so stiff that the deflection at the midpoint between U-frames, developed in Reference 3, is sufficiently small.

[6] A steel fabricator has assured me that he can produce the U-frames with the indicated surfaces accurate to about $\pm 0.005''$ using standard methods.

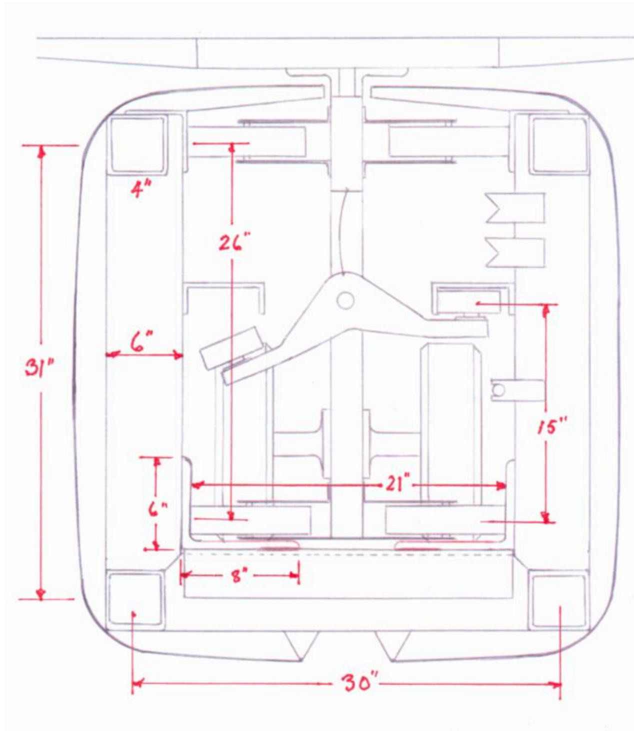


Figure 2. The Guideway Cross Section

3) The reaction surfaces for the LIMs are to be covered with 0.080" thick copper sheets, which are 10" wide and are wrapped around to the underside of the angles.
4) Upper running surfaces for 8" OD upper lateral polyurethane tires consist of 4x4x1/4" steel angles. They are secured to the upper inside edge of the U-frames.[7]
5) Four longitudinal stringers consisting of 4" OD hollow steel tubes welded to the outsides of the U-frames.
6) Diagonal truss members welded to the U-frame members to provide the necessary stiffness to the guideway. As developed in Appendix A, these members are placed in the direction that will almost always put them in tension rather than compression.
7) Switch rails welded to the U-frames at the merge and diverge sections of the guideway with entry and exit flaring of a length determined by dynamic analysis of the lateral motion of the vehicle into and out of each of these sections as developed in Appendix D. The flare shape is a quadratic, i.e., an equation of the form $y=kx^2$. The length of the flare is proportional to line speed.
8) Power rails that are to carry 600-volt d. c. current for vehicle propulsion.
9) Lossy cables for communication between the vehicles and the wayside zone controllers.
10) Composite covers lined with an aluminum spray.

To provide adequate stiffness to the guideway in both bending and torsion, the guideway is clamped to each support post via a bracket that resists bending and twisting moments.[8] Expansion joints are placed at the 20% point in each span where the bending moment is very small. Each span will typically be 90 feet or

[7] This angle has a cross sectional moment of inertia of 3.00 in^4. The maximum deflection under a point load $P = 250$ lb (see Section 5) at the center, assuming the beam is clamped at the ends, is $PL^3/192EI = 0.0023''$, which is not enough to cause ride-comfort problems.

[8] This idea was first discussed by Dr. Jack H. Irving, op. cit. page 219.

1080 inches in length, but may vary to avoid underground utilities. The U-frames, stringers and diagonals of a section of guideway between a pair of posts are illustrated in Figure 1. The support posts are octagonal and tapered from 10” at the top to 20” at the base and are fabricated out of 5/16” steel plate, typically 16 ft long, but will vary to conform to the topology and to local code.

3. Loads and Dimensions[9]

- ✓ The design is based on a vertical load of fully loaded vehicles nose-to-tail on the guideway.[10] The 9-ft-long vehicles r.m.s.[11] loaded weigh 1500 lb, giving a uniform load of 14 lb/in.
- ✓ A fully loaded vehicle will weigh W_v = 2100 lb.
- ✓ With vehicles nose-to-tail on the guideway an additional lateral load due to a 60-mph crosswind is assumed as the maximum side load.
- ✓ With no vehicles on the guideway, a 180-mph crosswind is assumed as the maximum side load.
- ✓ OD = 4” = Outside diameter of the four tube stringers. The required tube properties are given in Table 2.
- ✓ h = 31” = distance between centers of upper and lower stringers
- ✓ a = distance from centers of lower stringers to the neutral axis of the truss cross section.
- ✓ I_{nn} = moment of inertia of the guideway about its neutral axis.
- ✓ The U-frames are 6” x 8.2 lb/ft channel sections with a flange width of 1.92” and a moment of inertia of 13.1 in^4.
- ✓ The lower running surfaces are 8 x 6 x 1/2” angles with the longer dimension horizontal, of weight 23.2 lb/ft and cross-sectional area 6.80 in^4.
- ✓ The upper running surfaces are 4 x 4 x ¼” angles, of weight 6.58 lb/ft and cross-sectional area 1.93 in^4.
- ✓ Wheel base of vehicle is 84”.

[9] The properties of all the steel shapes used in this paper are taken from the *Manual of Steel Construction*, American Institute of Steel Construction, Third Edition.
[10] See J. E. Anderson, *Transit Systems Theory*, Section 10.3 for the reasons this is the critical loading condition.
[11] Root Mean Square loading is used because the probability that every vehicle is fully loaded is very small.

- ✓ Distance between U-frames is 54”.
- ✓ $b =$ 36” = estimate of the vertical distance between the center of pressure of the wind force and the centerline of the upper lateral wheels.
- ✓ $c =$ 26” = distance between centerlines of the upper and lower lateral wheels.
- ✓ W_p = 500 lb = weight of test passenger, lb.
- ✓ $d =$ 20” = lateral off-set of test passenger from vertical centerline of vehicle.
- ✓ e = 2” = distance between center of lower lateral wheels and the running surface.
- ✓ $L_{sep} = 30$ in lateral separation between the centerlines of the tubular stringers.
- ✓ A_s = cross sectional area of each of the tube stringers

3. <u>The Neutral Axis for Bending in the Vertical Plane</u>

The area moments of the lower tubes and angles about the neutral axis is equal to the area moments of the upper tubes and angles about the neutral axis. Thus

$$\begin{aligned} &A_s a + 7.5(0.5)(a - 4.5) + 6(0.5)(a - 7.25) \\ &\qquad = A_s(h - a) + 3.75(0.25)(h - a + 0.125) + 4(0.25)(h - a + 2.125) \end{aligned}$$

or

$$a = \frac{h(A_s + 1.9375) + 40.867}{2A_s + 8.688}$$

5. <u>The Moment of Inertia for bending in the vertical plane</u>

$$\begin{aligned} I_{nn} = (0.8)(2)\Big\{&A_s[a^2 + (h - a)^2] + 7.5(0.5)(a - 4.5)^2 + \frac{1}{12}(0.5)6^3 + 6(0.5)(a - 7.25)^2 \\ &+ \frac{1}{12}(0.25)(3.75)^3 + 3.75(0.25)(h - a + 0.125)^2 + 4(0.25)(h - a + 2.125)^2\Big\} \\ &= 1.6\{A_s[a^2 + (h - a)^2] + 3.75(a - 4.5)^2 + 10.099 + 3(a - 7.25)^2 \\ &+ 0.9375(h - a + 0.125)^2 + (h - a + 2.125)^2\} \end{aligned}$$

The factor of 0.8 is included based on advice of Structures Professor Ted Galambos that this reduction is needed to calculate the actual stress in and deflection of the truss.

6. Design of the U-Frames

Initially the U-frames can be fabricated by welding three pieces of channel section, which would be cut at a 45-deg angle at the two lower corners. In quantity, these frames can be stamped out of steel plate. About 634 of them are needed for the test system.

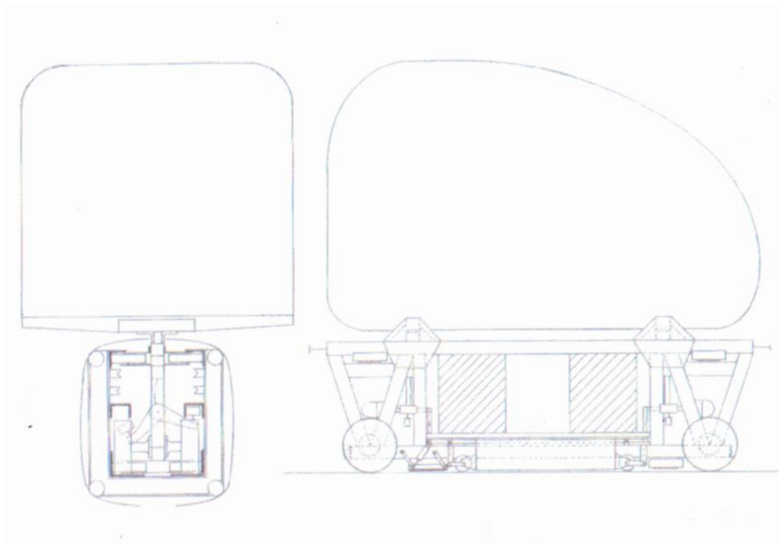


Figure 3. The vehicle mounted on the guideway.

The maximum force on the upper lateral wheel has been calculated by static analysis in the paper "The Maximum Side-Wheel Loads, and was found to be 1021 lb. Dynamic analysis developed in the paper "Lateral Dynamics of the ITNS Vehicle" found maximum values of the upper lateral wheel load to be 1876 lb. The lever arm to the lower inside corner of the U-frame is 27" and thus the maximum moment is (1876)(27) = 50,700 in-lb.

The maximum bending stress at the lower corner of the U-frame is

$$\sigma_{\max} = k\frac{Mc}{I}$$

in which k is the stress-concentration factor. For the U-frame channel given in Section 3 I = 13.1 in^4 [12] and c = 3". Then

$$\sigma_{max} = k\frac{(50{,}700\ inlb)(3\ in)}{13.1\ in^4} = 11{,}620k \text{ psi}$$

Mark's Standard Handbook for Mechanical Engineers, 10$^{\text{th}}$ Ed., page 5-5, Figure 5.1.6 IV gives k=2.8 for r/d = 0.01, where r = 0.085" and d = 6 x sqrt(2) = 8.49". Then maximum stress = 32,540 psi, which requires high-strength steel. While stress concentration is very important at a sharp corner, the plan will be to weld the U-frames to both the lower tube stringer and the lower angle, which will substantially reinforce the corner of the U-frame. Finite-element analysis will

[12] Per *Manual of Steel Construction* AISC, p. 1-31, this is the I for a 6" x 8.2lb/ft channel and c = 3".

be performed, and to be sure a U-frame will be loaded to destruction to prove the design. This analysis will also determine the type of steel to be used, likely stronger than mild steel. In any case, I will assume that the above-described channel will be adequate and that the sharp corner, the way it will be reinforced, will not be a problem.

The deflection of the U-frame under the above-calculated maximum load is

$$\Delta = \frac{F_u(h+2.25)^3}{3EI} = \frac{(1876)(33.25)^3}{3(29.5)(10)^6(13.1)} = 0.059''$$

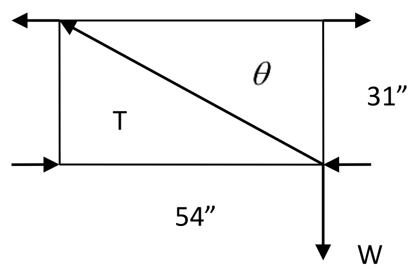


Figure 4. The diagonal stiffener

7. <u>The Diagonal Stiffeners</u>

Figure 1 illustrates a section of the guideway. Assume the ends are fixed, as they will be since the guideway is clamped to each post. As shown in Figure 2, the vertical dimension between the centerlines of the tube stringers is h = 31”. The horizontal distance between U-frames is 54”. The angle θ shown in Figure 4 is $\theta = tan^{-1}(54/31) = 60.14^o$. The load W, shown downward on the right of a U-frame, is half the weight of the loaded guideway between the segment shown and a mirror-image segment to the left of the next post to the right. If the distance between support posts is (90)(12) = 1080” then the distance between the U-frame shown and the mirror image projected from the next post is 1080 – 54 = 1026”.

The guideway weight is calculated in Section 8, but we must use it here to estimate the required cross sectional area of the diagonal stringers. The most severe loading condition has fully loaded vehicles nose-to-tail along the entire span. This gives a load of 14 lb/in. The maximum guideway weight, calculated in Section 8, is 14.0 lb/in. Thus we take the maximum load W on a 54” section of guideway closest to a support post as

$$W = \left(28\frac{lb}{in}\right)\left(\frac{1026\ in}{2}\right) = 14{,}364\text{ lb}$$

But this load is resisted by a pair of diagonal stringers. Thus, from Figure 4, the tension in each diagonal is $T = W/2\cos\theta = 1.004W$ = 14,426 lb. The diagonal members are selected to be in tension and we assume a standard design stress of 15,000 psi. Thus, the cross sectional area of the diagonal member needs to be at least 0.962 in^2. The length of the diagonal stiffener is $\sqrt{54^2 + 31^2} = 62.27"$. An open section, such as an angle, is best for the diagonal stringers.

This angle can be welded to the U-frames with the vertical side on the outside of each U-frame as shown in Figure 5. A 2x2x1/4" angle has a cross sectional area of 1.07 in^2 and weighs 3.65 lb/ft. With a length of 62.3" and with steel weighing 0.283 lb/in^3 the stringer weighs 18.9 lb or 18.9/54*12 = 4.2 lb/ft of guideway. In a clamped beam under uniform load shear is maximum at the ends and decreases linearly to zero at the center of the span. Thus under uniform load the maximum load the diagonal stiffeners must carry decreases linearly to zero at the center of the span, and hence we should specify smaller angles as we approach the center.

The detail of placement and welding of the diagonal is shown in Figure 5. In this drawing it is assumed that the flat side of each of the two C-section U-frames shown is to the left. The angle is to be cut so that it can lie flat against the side of each of the C-section U-frames. The lower-right end of the diagonal can be cut straight across to provide a weld surface on its horizontal side, and the upper-left end can be cut to follow the curve of the upper tubular stringer to provide a weld surface there. Assume conservatively that the entire load carried by the welds is in shear. The shear strength is half the tensile strength, so assume the design shear strength to be 7500 psi. Then with the maximum diagonal tensile load of 14,500 lb, the required weld area is 14,500/7500 = 1.93 in^2. Since the weld length is 9.2 inches, the weld width must be at least 1.93/9.2 = 0.21 in.

In some cases, discussed in Appendix A, the load will be asymmetric so some of the diagonal stiffeners will be occasionally in compression, in which cases we must consider buckling. The Euler buckling load is

$$P_{cr} = \alpha\pi\frac{EI}{l^2}$$

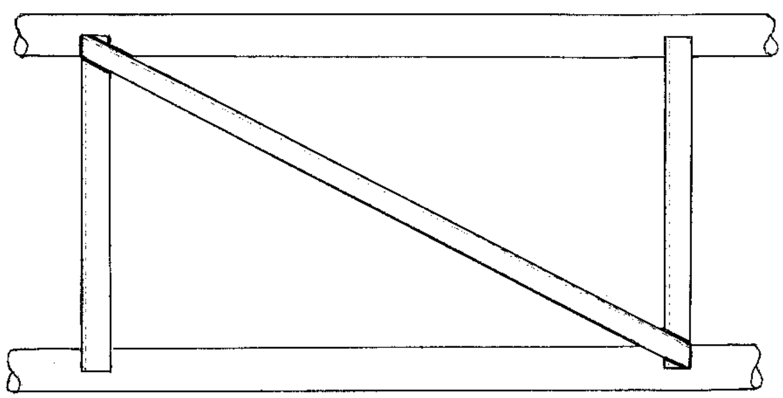


Figure 5. Diagonal detail showing diagonal weld area.

in which the dimensionless factor α is 1 for simply supported ends and 4 for clamped ends. Since the ends of our diagonals are welded in place, assume $\alpha = 4$. The smallest moment of inertia for a 2x2x1/4 angle is 0.141 in^4. If the angle buckles under a compressive load, it will buckle along this axis. The length l of the diagonal as calculated above is 62.27 in. Thus for this angle,

$$P_{cr} = 4\pi \frac{29.5(10)^6(0.141)}{62.27^2} = 13{,}480\ lb$$

A possible distribution of the diagonals with their properties may be as given in Table 1, where the average angle weight per foot is given. Assuming the same distribution of angle stiffeners in the bottom of the guideway as in the sides, with the inclination of the bottom stiffeners alternating in a zigzag pattern, the weight of diagonal stiffeners per lineal inch of guideway is

$$\frac{2.43\ lb/ft}{12\ in/ft}\left[2\sqrt{1+\left(\frac{31}{54}\right)^2} + \sqrt{1+\left(\frac{30}{54}\right)^2}\right] = 0.699\ lb/in$$

We see that in all cases, the maximum tension in each diagonal is less than the buckling load. Moreover, it is shown in Appendix A that in no loading case or vehicle position will the compression in any of the angles be close to the buckling load.

Table 1. Properties of Diagonal Stiffeners

Span	1080	in
Maximum load on guideway	14	lb/in
Guideway weight	14	lb/in
Vertical distance between stringers	31	in
Longitudinal distance between U-Frames	54	in
Length of stiffener	62.27	in
Angle, vertical to diagonal	1.050	radians
Design stress	15,000	psi
Modulus of Elasticity	29,500,000	psi

Gdwy Section	Length Supported in	Maximum Load lb	Tension in Stiffener lb	Required Area in^2	Commercially available angle	Area of Angle in^2	Weight lb/ft	Radius of Gyration in	Moment of Inertia in^4	Buckling Load lb
1	1026	14364	14426	0.962	L2x2x3/8	1.37	4.65	0.386	0.2041	19518
2	918	12852	12907	0.860	L2x2x1/4	0.944	3.21	0.387	0.1414	13519
3	810	11340	11389	0.759	L2x2x1/4	0.944	3.21	0.387	0.1414	13519
4	702	9828	9870	0.658	L2x2x3/16	0.722	2.46	0.389	0.1093	10447
5	594	8316	8352	0.557	L2x2x3/16	0.722	2.46	0.389	0.1093	10447
6	486	6804	6833	0.456	L2x2x1/8	0.491	1.67	0.391	0.0751	7177
7	378	5292	5315	0.354	L2x2x1/8	0.491	1.67	0.391	0.0751	7177
8	270	3780	3796	0.253	L2x2x1/8	0.491	1.67	0.391	0.0751	7177

9	162	2268	2278	0.152	L2x2x1/8	0.491	1.67	0.391	0.0751	7177
10	54	756	759	0.051	L2x2x1/8	0.491	1.67	0.391	0.0751	7177
						Average:	2.43			

8. <u>The guideway weight per unit of length</u>

Weight of 4 Stringers

1	2	3	4
4" O.D. x 0.174 wall	4" O.D. x 0.233 wall	4" O.D. x 0.291 wall	4" O.D. x 0.349 wall
37.6	48.8	59.2	68.8
lb/ft	lb/ft	lb/ft	lb/ft

Upper and lower angle running surfaces1: 2(23.2 + 6.58) = 59.6 lb/ft.

Weight of U-frame = 8.2 lb/ft (2 h +26)/12 = 8.2(88)/12 = 60.2 lb.

Weight of U-frames per foot of guideway = 60.2/4.5 = 13.4 lb/ft.

Weight of diagonals = 0.699 lb/inx12 = 8.40 lb/ft of guideway length.

Weight of guideway covers and brackets = $126\,\text{in} \times 3/16\,\text{in} \times 0.043\,\text{lb/in}^3 \times 12\,\text{in/ft} = 12.2$ lb/ft.

Weight of power rails = 6 lb/ft

Weight of copper reaction sheets = 20x0.080x12x0.33 lb/in^3 = 6.34 lb/ft

Total guideway weight = Stringer weight + 59.6 + 13.4 + 8.40 + 12.2 + 12.34 = Stringer weight + 106 lb/ft

9. <u>Maximum deflection of loaded guideway</u>

The guideway is clamped at the posts. Therefore, the maximum deflection is

$$\Delta_{max} = \frac{(w_g + w_v) Span^4}{384 E I_{nn}}$$

where, assuming fully loaded vehicles nose to tail, the vehicle weight is 14 lb/in.

10. <u>Maximum bending stress due to vertical load</u>

$$\sigma_{\max} = \frac{Mc}{I_{nn}}, \quad M = \frac{wL^2}{12}, \quad c = h - a, \quad \sigma_{\max} = \frac{(w_g + w_v) L^2 (h-a)}{12 I_{nn}}$$

11. Maximum lateral wind loading on the unloaded guideway per unit length

$$F_{wind} = \frac{\rho g}{2g} V_{wind}^2 C_D (h + 6\ in)$$

where $\rho g = 0.075\,\text{lb/ft}^3, V_{wind} = 180\,\text{mph} = 264\,\text{ft/sec}, C_D = 0.8$. This drag coefficient assumes that the guideway is covered with 6" radii at the top and bottom of the covers.[13] Thus

$$F_{wind} = \frac{0.075 \frac{lb}{ft^3} (264 ft/sec)^2 (0.8)(31 + 6\ in)}{64.4 \frac{ft}{sec^2}} \left(\frac{1\ ft}{12\ in}\right)^2 = 16.7\ lb/in$$

12. Lateral Moment of Inertia of Guideway

The moment of inertia against a side-wind load is

$$\begin{aligned} I_{yy} = (0.8)(2) \Bigg[& 2A_s(15)^2 + 6\left(\frac{1}{2}\right)\left(\frac{21.5}{2}\right)^2 + \frac{1}{12}(0.5)(7.5)^3 + 7.5(0.5)\left(\frac{21 - 7.5}{2}\right)^2 \\ & + 4(0.25)\left(13 - \frac{1}{8}\right) + \frac{1}{12}(0.25)(3.75)^3 + (3.75)(0.25)15^2 \Bigg] \\ & = 1.6(450A_s + 346.7 + 17.6 + 170.9 + 12.9 + 1.1 + 211) = 720A_s + 1216 \end{aligned}$$

13. Bending stress due to side wind

$$\sigma_{max} = \frac{Mc}{I_{yy}} = \frac{F_{wind} L^2}{12 I_{yy}} \frac{37in}{2} psi$$

where L is the span, typically 1080 in.

14. Numerical results related to guideway bending

The above equations are solved in an Excel program with the following results. Note that ASHTO standards permit a ratio of span to maximum deflection of no less than 800:1 and APM Standards require no less than 1000:1. These results show that the smallest wall thickness, 0.116 in, is satisfactory for straight guideways, and that the choice between square and round tubes can be made by the fabricator. Additional analysis is needed to determine the guideway properties in torsion and in curves.

[13] C. Scraton and E. W. E. Rodgers, "Steady and Unsteady Wind Loading," Phil. Trans. Roy. Soc. London. A. 269(1971) 353-379.

Table 2a. Properties of the ITNS Guideway with Square-Tube Stringers

Modulus of Elasticity	29,500,000	29,500,000	29,500,000	29,500,000	29,500,000	psi
Span Length	1080	1080	1080	1080	1080	in
Tube Stringer OD	4	4	4	4	4	in
Tube Wall Thickness	0.116	0.174	0.233	0.291	0.349	in
Tube Cross Sectional Area	1.77	2.58	3.37	4.1	4.78	in^2
Tube Weight	6.46	9.40	12.2	14.8	17.2	lb/ft
h	31	31	31	31	31	in
a	12.74	13.06	13.31	13.50	13.65	in
Guideway Weight	132	144	155	165	175	lb/ft
Vehicle RMS Gross Weight	168	168	168	168	168	lb/ft
Maximum Wind Speed	180	180	180	180	180	mph
Maximum Operational Wind Speed	60	60	60	60	60	mph
Sidewind Loading	230	230	230	230	230	lb/ft
Ivertical	3144	3784	4405	4976	5507	in^4
Ihorizontal	2490	3074	3642	4168	4658	in^4
Maximum Bending Stress:						
Vertical	14105	11963	10498	9490	8747	psi
Horizontal	13849	11221	9469	8275	7405	psi
Maximum Vertical Deflection	0.954	0.824	0.733	0.670	0.623	in
Span/Maximum Deflection	1132	1310	1473	1612	1734	

Table 2b. Properties of the ITNS Guideway with Round-Tube Stringers

Modulus of Elasticity	29,500,000	29,500,000	29,500,000	29,500,000	29,500,000	psi
Span Length	1080	1080	1080	1080	1080	in
Tube Stringer OD	4	4	4	4	4	in
Tube Wall Thickness	0.116	0.174	0.233	0.291	0.349	in
Tube Cross Sectional Area	1.42	2.09	2.76	3.39	4.00	in^2
Tube Weight	4.82	7.13	9.40	11.56	13.64	lb/ft
h	31	31	31	31	31	in

a	12.57	12.88	13.12	13.32	13.48	in
Guideway Weight	125	135	144	152	161	lb/ft
Vehicle RMS Gross Weight	168	168	168	168	168	lb/ft
Maximum Wind Speed	180	180	180	180	180	mph
Maximum Operational Wind Speed	60	60	60	60	60	mph
Sidewind Loading	230	230	230	230	230	lb/ft
Ivertical	2862	3399	3924	4421	4901	in^4
Ihorizontal	2235	2722	3201	3657	4098	in^4
Maximum Bending Stress:						
Vertical	15296	13065	11498	10373	9515	psi
Horizontal	15431	12672	10774	9430	8416	psi
Maximum Vertical Deflection	1.026	0.891	0.795	0.725	0.671	in
Span/Maximum Deflection	1053	1212	1359	1490	1609	

15. Twist of the guideway due to a sidewise moment

If θ is the angle of twist at a certain point *x* along the guideway, measured from one of the support posts, the twist per unit length is[14]

$$\frac{d\theta}{dx} = \frac{M(x)}{GI_p}$$

Assume the guideway is subjected to a moment *m* per unit of length, so that the total moment on a guideway of length *L* is *M* = *mL*. Then at a point a distance *x* from one of the posts, from free-body considerations the moment is $M(x) = m\left(\frac{1}{2}L - x\right)$ so that

$$\frac{d\theta}{dx} = \frac{m\left(\frac{1}{2}L - x\right)}{GI_p}$$

Assume the guideway is clamped at the posts so that $\theta = 0\ at\ \mathrm{x} = 0\ and\ x = L$. Then, integrating,

$$\theta = \frac{m(Lx - x^2)}{2GI_p}$$

Thus, the maximum twist, at $x = \frac{1}{2}L$ is

[14] S. Timoshenko, *Strength of Materials, Part I and Part II, 2nd Ed.,* D. Van Nostrand and Company, 1940.

$$\theta_{max} = \frac{mL^2}{8GI_p}$$

From Section 6 the maximum side force on each upper wheel is 1876 lb, and the moment arm is 27". Since the U-frames are 54 inches apart, the moment per unit length is m = (1876)(27)/54 = 938 in-lb/in. The polar moment of inertia is given[15] for four tube sizes. G = 29,500,000/2.6 = 11,350,000 psi. Hence with a span of $L = 90(12) = 1080$" we obtain the following maximum twist angles, varied by the OD and wall thickness of the stringers.

Stringer Dimensions	4"ODx0.174" wall	4"ODx0.233" wall	4"ODx0.291" wall	4"ODx0.349" wall
I_p, in^4	27.0	33.2	41.4	44.93
θ_{max}, deg	9.5	7.8	6.2	5.7

16. <u>Deflection of a Curved Guideway</u>

This analysis is carried out in Appendix B where the maximum deflection at the center point of a span is calculated for a range of span lengths and guideway curve radii, with conclusions given there.

17. <u>The Guideway Natural Frequency</u>

From *Marks' Standard Handbook for Mechanical Engineers*, 10th Ed., page 3-73 the fundamental natural frequency of a beam clamped at both ends is

$$f_1 = \frac{1}{2\pi}(1.506\pi)^2\sqrt{\frac{EIg}{wL^4}} = 3.563\sqrt{\frac{EIg}{wL^4}}$$

in which the modulus of elasticity for steel is $E = 29.5(10)^6$psi and g = 32.174 ft/sec^2. For span length L= 90 ft, we obtain the following values for the fundamental natural frequency for vertical motion.

Stringer	4"ODx0.174" wall	4"ODx0.233" wall	4"ODx0.291" wall	4"ODx0.349 wall
Moment of Inertia, in^4	3784	4405	4976	5507
Guideway weight, lb/in	12.0	12.9	13.75	14.58

[15] See the internal paper "The Polar Moment of Inertia of the ITNS Guideway"

$\sqrt{\frac{I}{w}}$	17.8	18.48	19.02	19.44
f_1, Hz	5.80	6.02	6.20	6.34

With 60-ft spans the natural frequencies will be higher by the ratio $(90/60)^2 = 2.25$. The forcing frequency is due to the motion of vehicles across the span. With vehicles moving half a second apart, the forcing frequency would be 2 Hz, which is only a third of the fundamental natural frequency, which means, from the theory of vibrations, that amplification due to passage of vehicles will be small. Moreover, since we use asynchronous control, the spacings of vehicles will not be uniform, which will cause damping of any vibrations. The phenomenon is similar to commanding a company of men to proceed out of step when crossing a bridge to void destructive amplifications.

18. <u>The Critical Speed</u>

The maximum deflection in a beam clamped at both ends, which is the way this guideway will be mounted, is

$$\Delta_{max} = \frac{wL^4}{384EI}$$

in which w is the weight of the loaded guideway per unit of length, L is the distance between posts, E is the modulus of elasticity, and I is the guideway vertical moment of inertia. The vertical motion of the passenger will be

$$y(t) = \frac{1}{2}\Delta_{max} sin(2\pi f t)$$

in which $f = V/L$ is the frequency of motion, V is the speed, and t is time. Differentiating twice, the maximum vertical acceleration experienced by the passengers is

$$a_{max}(f) = 2\pi^2 \Delta_{max} f^2$$

in which, in the region of interest, we will find in Appendix C that the maximum comfort acceleration is a function of f. For the conditions given in Appendix C, we find that the critical speed for the guideway properties listed in Table 2 with the smallest stringers is 130 mph, and for the largest stringers 159 mph.

These results assume that the guideway is built flat. Suppose, however, that each span is built with camber such that when the guideway rests on the support posts under the force of its weight it lies flat. Then suppose a vehicle of gross weight W moves at speed V across the beam. The maximum deflection of a clamped beam under a central point load W is

$$\Delta_{max} = \frac{WL^3}{192EI} = \frac{2W}{wL} \times \frac{wL^4}{384EI}$$

Using the data that produced Table 2, the factor *2W/wL* = 2(2100)/132/90 =0.354, and, from the equation for V_{cr} given in Appendix C, the critical speed could be increased by a factor up to $0.354^{-0.4} = 1.52$, which would be higher than any critical speed of interest. Thus the guideway specified by Table 2 does not impose a speed limit. The more likely cause of a speed limit results from the fact that the thrust required to overcome air drag is proportional to the square of speed, which, for the same guideway will require motors larger in proportion to the increased thrust.

The practical conclusion is that to move at speeds upward of say 100 mph, higher-powered vehicles are needed, which divides the problem into two classes of vehicles: one set for urban speeds and a second set for inter-city speeds. Both, however, can operate on the same guideway.

Appendix A. Compression in the Diagonals

It is important to determine if there will be any condition in which some of the diagonals could be subject to high enough compressive loads to buckle. Knowing the sign of shear on the guideway cross section under uniform load, we have placed the diagonals in the direction that will place them in tension when the load is uniform. We need to determine if there can be loading conditions that may place some of the diagonals in compression. To set the ground work, it is necessary to first determine the magnitude and distribution of shear when the guideway is under uniform load.

<u>The Guideway with a Uniform Load Only</u>

Consider a beam of length L clamped at both ends and subject to a uniform load *w* lb/in. The vertical reaction force at each end is $R = wL/2.$ The moment M must be applied at each end to force the slope there to zero. Let *x* be the coordinate along the beam with $x = 0$ at the left end and $x = L$ at the right end. Let $y(x)$ be the deflection of the beam. The differential equation for beam deflection is

$$EI\frac{d^2y}{dx^2} = M(x) = M - Rx + \frac{wx^2}{2}$$

in which E is the modulus of elasticity and I is the moment of inertia. Integrating once, the slope is given by

$$EI\frac{dy}{dx} = Mx - \frac{wL}{2}\frac{x^2}{2} + \frac{wx^3}{6}$$

Because of symmetry, the slope is zero at the center of the beam. Thus

$$M = \frac{wL}{2}\frac{L}{4} - w\frac{L^2}{24} = \frac{1}{12}wL^2.$$

Integrating once more, the deflection of the beam is

$$y = \frac{w}{EI}\left(\frac{L^2x^2}{24} - \frac{Lx^3}{12} + \frac{x^4}{24}\right) = \frac{wx^2}{24}\left(L^2 - 2Lx + x^2\right)$$

from which the maximum deflection at the center $x = L/2$ is the well-known value

$$y_{\max} = \frac{wL^4}{384EI}$$

Note that the bending moment can now be written in the form

$$M(x) = \frac{wL^2}{12} - \frac{wL}{2}x + \frac{wx^2}{2}$$

We see that the values of *x* at which the moment is zero are solutions of the quadratic equation

$$x^2 - Lx + \frac{L^2}{6} = 0$$

Thus, the moment is zero at the points

$$\frac{x}{L} = \frac{1}{2}\left(1 \pm \sqrt{\frac{1}{3}}\right) = 0.211, 0.789$$

i.e., the moment is zero at the point 21% from each end of the beam, which makes it the logical place for an expansion joint.

The shear force in the beam is the derivative of the moment. Thus

$$Shear(x) = w\left(-\frac{L}{2} + x\right)$$

Thus Shear goes from $-\frac{wL}{2}$ at the left end of the beam to 0 at the center to $+\frac{wL}{2}$ at the right end.

<u>The Guideway with a Point Load added to the Uniform Load</u>

Now consider the same beam subject to a point load P at a point *a* from the left end. In this case let the empty weight of the beam per unit of length be w_e. Also, in this case, the left and right end moments M_L, M_R are in general different and the left and right end reactions R_L, R_R are in general different. The differential equation for the beam is

$$EI\frac{d^2y}{dx^2} = M(x) = M_L - R_L x + \frac{w_e x^2}{2} + P\langle x - a\rangle$$

in which the notation $\langle x - a\rangle = x - a \; if \; x \geq a$ but $0 \; if \; x < a$. Integrating once

$$EI\frac{dy}{dx} = M_L x - R_L\frac{x^2}{2} + \frac{w_e x^3}{6} + \frac{P\langle x-a\rangle^2}{2}$$

Since the slope is zero at the right end of the beam, we have

$$M_L = \frac{R_L L}{2} - \frac{w_e L^2}{6} - \frac{P(L-a)^2}{2L}$$

Integrating once more we have

$$EIy = M_L\frac{x^2}{2} - R_L\frac{x^3}{6} + w_e\frac{x^4}{24} + P\frac{\langle x-a\rangle^3}{6}$$

Since the deflection is zero at the right end, we have a second equation for M_L. Thus

$$M_L = \frac{R_L L}{3} - \frac{w_e L^2}{12} - \frac{P(L-a)^3}{3L^2}$$

Equating the two formulae for M_L and multiplying by 6/*L* we get

$$R_L = \frac{w_e L}{2} + P\left(1-\frac{a}{L}\right)^2\left(1+2\frac{a}{L}\right)$$

The shear in the beam is

$$S(x,a) = \frac{dM(x)}{dx} = -R_L + w_e x + P\frac{d}{dx}\langle x-a\rangle$$

where $\frac{d}{dx}\langle x-a\rangle = 1$ if $x > a,$ otherwise 0. Substituting for R_L

$$S(x,a) = -\frac{w_e L}{2}\left(1-\frac{2x}{L}\right) - P\left[A - \frac{d}{dx}\langle x-a\rangle\right]$$

where

$$A = \left(1-\frac{a}{L}\right)^2\left(1+2\frac{a}{L}\right) = \left(1-2\frac{a}{L}+\frac{a^2}{L^2}\right)\left(1+2\frac{a}{L}\right) = 1-3\frac{a^2}{L^2}+2\frac{a^3}{L^3} = 1-\frac{a^2}{L^2}\left(3-2\frac{a}{L}\right)$$

Note that $A = 1$ when $a = 0, A = \frac{5}{32}$ when $a = L/4$ and $A = \frac{1}{2}$ when $a = \frac{L}{2}$.
If $x > a$

$$S(x,a) = -\frac{w_e L}{2}\left(1-\frac{2x}{L}\right) + P(1-A) = -\frac{w_e L}{2}\left(1-\frac{2x}{L}\right) + P\frac{a^2}{L^2}\left(3-2\frac{a}{L}\right)$$

Thus, at $x/L = ½$, S is positive at the center of the guideway. This means that one or more of the diagonals will be in compression. With one vehicle at the center of the span, $a/L = ½$, so positive shear there is equal to one half the vehicle weight. Now suppose this vehicle is trailed at one minimum headway by a vehicle of the same weight. Suppose the speed is 30 mph or 44 ft/sec and the minimum headway is 0.5 sec. Then the second vehicle would be 22 ft behind the first. With a span of 90 ft, this is close to a quarter of the span, implying $a/L = ¼$, in which case $1 - A = 5/32$. So with these two vehicles on the span, the maximum shear at the center would be $P(1/2 + 5/32) = 0.656P$. With a vehicle gross weight of 2100 lb, the maximum shear would be 0.656(2100) = 1378 lb. As shown in Section 7, the compressive force in the diagonal would be greater by the factor $\frac{\sqrt{54^2+31^2}}{31} = 2.009$. Thus, the maximum compressive force in the diagonal brace would be (2.009)(1378) = 2768 lb, which compares with the buckling load given in Table 1, Section 7 of 7177 lb. Since this calculation gives the highest compressive force in any of the diagonals, it is clear that none of the diagonals will buckle.

The Guideway-Support Posts

We have specified that the support posts be thin-walled, octagonal, tapered steel. The preferred manufacturer is Millerbernd of Winsted, Minnesota. With their equipment, it is much less expensive to manufacture octagonal posts rather than round posts. In this paper, we develop their properties.

<u>Cross sectional Moment of Inertia of a Solid Octagonal Cross Section</u>

Let the post diameter be *D* and the length of each of the eight equal sides be *s*. Then,

$$s = \frac{D}{1+\sqrt{2}}\left(\frac{1-\sqrt{2}}{1-\sqrt{2}}\right) = D\left(\sqrt{2}-1\right), \qquad s^2 = D^2\left(3-2\sqrt{2}\right), \qquad s^4 = D^4\left(17-12\sqrt{2}\right)$$

Establish an x-y coordinate system with origin at the center of the post. Then consider the first quadrant of the cross section, which is divided into three parts. Thus

$$\begin{aligned}\frac{I}{4} = \int_0^{D/2} y^2\left(\frac{s}{2}\right)dy + \int_0^{\frac{s}{2}} y^2\left(\frac{D}{2}-\frac{s}{2}\right)dy + \int_{\frac{s}{2}}^{\frac{D}{2}} y^2\left(\frac{D}{2}-y\right)dy \\ = \frac{s}{6}\left(\frac{D}{2}\right)^3 + \frac{1}{6}(D-s)\left(\frac{s}{2}\right)^3 + \frac{D}{6}\left[\left(\frac{D}{2}\right)^3 - \left(\frac{s}{2}\right)^3\right] - \frac{1}{4}\left[\left(\frac{D}{2}\right)^4 - \left(\frac{s}{2}\right)^4\right] \\ = \frac{1}{48}\left[sD^3 + Ds^3 - s^4 + D^4 - Ds^3 - \frac{48}{64}\left(D^4 - s^4\right)\right] = \frac{1}{48}\left(\frac{D^4}{4} + sD^3 - \frac{s^4}{4}\right)\end{aligned}$$

$$I = \frac{D^4}{48}\left(1 + 4\sqrt{2} - 4 - 17 + 12\sqrt{2}\right) = \frac{\left(4\sqrt{2}-5\right)}{12}D^4$$

The moment of inertia of the cross section of a hollow cross section of material thickness *t*, is

$$I = \frac{\left(4\sqrt{2}-5\right)}{12}\left[D^4 - (D-2t)^4\right]$$

<u>Bending stress due to Side Wind</u>

The bending stress is given by the well-known formula

$$\sigma = \frac{Mc}{I}$$

where $c = \frac{D}{2}$. Thus, for our post

$$\sigma = \frac{M}{0.876tD^2}$$

We have determined that the value of D at the base of the post should be 24 inches, and that t = 5/16 in. With $\sigma = 15{,}000\ psi$, the maximum moment is

$$M = (15{,}000\ psi)(0.876)(\tfrac{5}{16}")(24)^2\,\tfrac{1\,ft}{12\,in} = 197{,}000 \text{ ft-lb.}$$

Assuming 17.5 ft to the center of the guideway, the maximum wind force per foot on a 90-ft section of guideway is

$$\frac{197{,}000}{17.5(90)} = 125\,\frac{lb}{ft}.$$

The wind force on a guideway of depth H per foot is given by

$$F_{wind} = \frac{\rho g}{2g} C_D H V_{wind}^2 = \frac{(0.075\,\frac{lb}{ft^3})}{64.4\ ft/sec^2}(0.6)(3.25\ ft)V_{wind}^2 = \frac{V_{wind}^2}{440} = 125$$

$$V_{wind_{max}} = \sqrt{(125)(440)} = 235\ ft/\sec = 160\ mph$$

The specified maximum wind in Chicago was 100 mph and in Florida 120 mph.

<u>Bending Stress due to Vertical Guideway Loading</u>

The bending moment at the support of a uniformly loaded beam is

$$M = \frac{1}{12}wL^2$$

Where L = 90 ft is the span and w = 150 + 122 = 272 lb/ft. Thus

$$M = \frac{272\times 90^2}{12} = 183{,}600 \text{ ft-lb,}$$

which is 7% less than the 197,000 ft-lb that would give a maximum stress of 15,000 psi.

<u>Shear Stress due to Twisting of the Post</u>

The twisting moment at a post due to side wind is

$$M_{twist} = \frac{w_{wind} L^2}{12}$$

With a side wind of 120 mph or 176 ft/sec,

$$w_{wind} = \frac{(176)^2}{440} = 70.4\ lb/ft$$

Thus, with L = 90 ft

$$M_{twist} = \frac{70.4(90)^2}{12} = 47{,}520 \text{ ft-lb.}$$

$$ShearStress \times Area \times Radius = M_{twist}$$

$$ShearStress = \frac{47{,}520 \times 12inlb}{8st \times D/2} = \frac{570{,}000\left(1+\sqrt{2}\right)}{4(5/16)D^2} = \frac{1{,}101{,}000}{D^2}$$

Assume the taper in the post is 2:1, i.e., D = 12 in at the top of the post. Then the maximum shear stress is 7,646 psi, which is acceptable for mild steel.

<u>Savings in Steel with Tapered Post</u>

$$VolumeSaved = \frac{8tL}{1+\sqrt{2}}\left[D(0) - \frac{3}{4}D(0)\right] = \frac{2tLD(0)}{\left(1+\sqrt{2}\right)} = \frac{2(5/16)(16 \times 12)(24)}{1+\sqrt{2}}$$
$$= 1193in^3 \times \frac{0.284lb}{in^3} = 339lb.$$

<u>The Deflection of a Tapered Post</u>

The moment equation for a tapered column of length L under a point load P at the end is

$$EIy" = E\left\{\frac{\left(4\sqrt{2}-5\right)}{12}\left[D^4 - (D-2t)^4\right]\right\}\frac{d^2y}{dx^2} = P(L-x) = PL\left(1 - \frac{x}{L}\right)$$

Let

$$\alpha = \frac{4\sqrt{2}-5}{12}, \quad D^4 - (D-2t)^4 = 8tD^3 - 24t^2D^2 + 32t^3D - 16t^4 \approx 8tD^2(D-3t)$$

in which for our post

$$D(x) = D(0) - [D(0) - D(L)]\frac{x}{L} = D(0)\left\{1 - \left[1 - \frac{D(L)}{D(0)}\right]\frac{x}{L}\right\}, \qquad \frac{D(L)}{D(0)} = \frac{1}{2}$$

Thus

$$D(x) = D(0)\left(1 - \frac{x}{2L}\right)$$

Let $u = 1 - \frac{x}{2L}, du = -\frac{dx}{2L}, D_o = D(0)$. Then

$$8\alpha t E D_o^3 u^2 \left(u - \frac{3t}{D_o}\right)\frac{1}{4L^2}\frac{d^2y}{du^2} = PL[1 - 2(1-u)]$$

Let

$$k = \frac{PL^3}{2\alpha t E D_o^3}, \qquad \beta = \frac{3t}{D_o}$$

Then

$$\frac{d^2y}{du^2} = k\frac{(2u-1)}{u^3 - \beta u^2} = k\left[\frac{2}{u(u-\beta)} - \frac{1}{u^2(u-\beta)}\right] \cong k\left(\frac{2}{u^2} - \frac{1}{u^3}\right)$$

Then

$$\frac{dy}{du} = k\left(-\frac{2}{u} + \frac{1}{2u^2}\right) + C_1$$

When $x = 0, u = 1$ and $\frac{dy}{du} = 0$. Thus

$$C_1 = k\left(2 - \frac{1}{2}\right) = \frac{3}{2}k, \qquad \frac{dy}{du} = k\left(\frac{3}{2} - \frac{2}{u} + \frac{1}{2u^2}\right)$$

$$y = k\left[\frac{3}{2} - 2ln(u) - \frac{1}{2u}\right] + C_2$$

When $u = 1, y = 0$. Thus

$$C_2 = -k$$

Thus,

$$y = \frac{k}{2}\left[1 - \frac{1}{u} - 4ln(u)\right]$$

When $x = L, u = \frac{1}{2}$. Thus

$$y(L) = k[2ln(2) - 1] = 0.386k = \frac{0.386PL^3}{2\alpha tED_o^3} = 3.53\frac{PL^3}{EtD_o^3} = \frac{3.53(16 \times 12)^3 P}{29.5(10)^6(5/16)(24)^3}$$
$$= 0.000196P = 0.000196(125)(90) = 2.2 \text{ in.}$$

<u>The Base Plate</u>

Let the base of the post be welded to a 2-in-thick square steel plate 36 in on each side secured to its foundation with four bolts spaced 30 in apart. Then, to resist a bending moment of 200,000 ft-lb, each bolt will be subject to a normal force of

$$F_{bolt} = \frac{1}{2}\frac{200{,}000\ ftlb}{2.5\ ft} = 40{,}000\ lb$$

For A307 steel bolts a tolerable design stress is 30,000 psi. Thus the root diameter *d* of each bolt must be at least

$$\pi\frac{d^2}{4} = \frac{40{,}000}{30{,}000}, d \geq 1.4 \text{ in}$$

The maximum shear stress in the base plate must be determined by finite-element analysis.

Ride Comfort in a Vehicle moving on a Flexible Surface

Abstract

This paper treats the problem of the requirements for ride comfort as a vehicle moves at constant speed along a flexible running surface that is supported at fixed, equal distances. The model used permits vertical and pitch motion, and compares with a standard comfort value the vertical acceleration of a passenger modeled as a point at its center of gravity and riding on a sprung and damped seat. The solution depends on thirteen parameters: the wave length and amplitude of deflection of the running surface, the wheelbase of the vehicle, its radius of gyration, the position of its center of gravity, the empty weight of the vehicle, the elasticity and damping of the tires, the speed, the horizontal position of the passenger, the weight of the passenger, and the passenger-seat spring constant and damping ratio. The equations of motion are solved analytically and results are presented for a useful range of parameters, thus providing information needed for design. With a midspan deflection of 0.055 in and reasonable passenger-seat stiffness, the vertical accelerations that will be experienced by the passenger are well below the comfort limit.

1. Introduction

A flexible running surface of an elevated ITNS system is supported at equal intervals λ along the direction of motion. Therefore, there will be a certain amount of up-down motion as a vehicle passes. The purpose of this analysis is to estimate the tolerable amount of deflection of the running surface as determined by the oscillating vertical acceleration felt by the passengers. The tolerable vertical acceleration is given in the following Figure 3-1, which is taken from the International Standards Organization. The stiffness k and damping ratio ζ of the support tires is taken into account and the passenger is supported by a seat with a given stiffness k_s and damping ratio ζ_s.

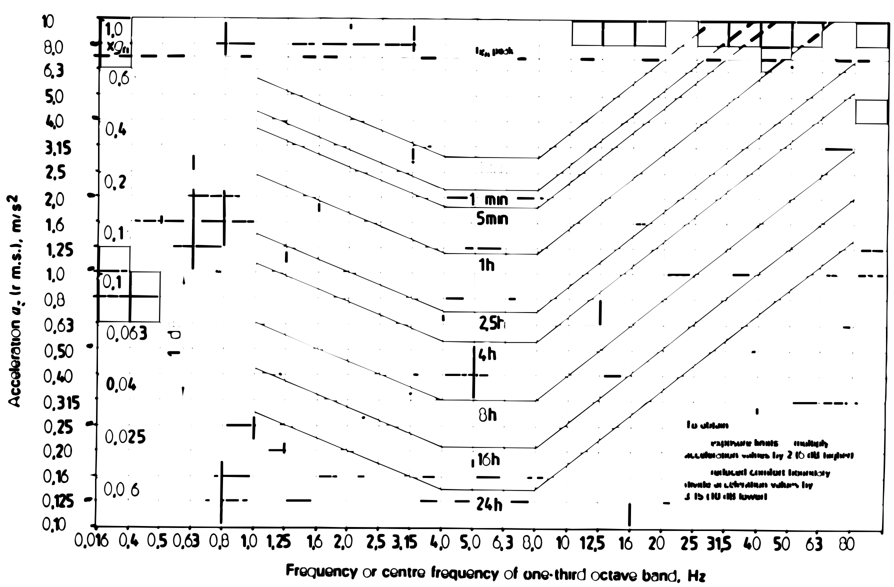


FIGURE 3-1 SPECTRAL COMPOSITION OF ACCELERATION

2. Problem Definition

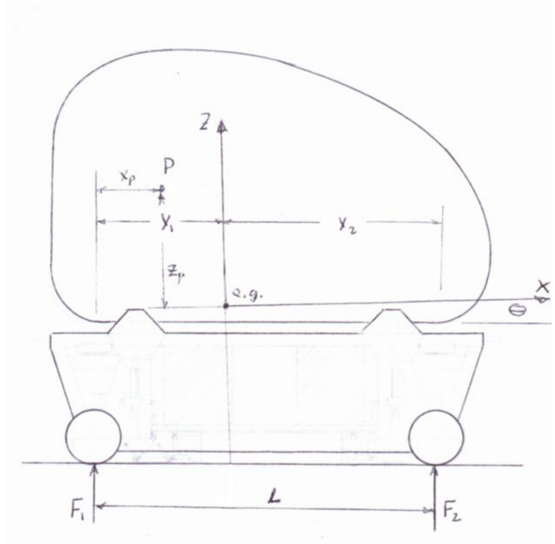


Figure 1. Notation used in analysis of a vehicle moving over a flexible guideway.

The notation used is shown in Figure 1. Assume the vehicle moves to the right as a rigid body at a speed V with its center of gravity (c. g.) measured from a fixed point $x = 0$ as

$$x = x_{cg} = Vt. \tag{1}$$

We take a z coordinate positive upward, and a pitch angle θ, positive in the counterclockwise direction. The z and θ coordinates relate to a smooth, flat, horizontal reference plane placed so that when $z = 0$ and $\theta = 0$ the undeflected tires just touch the reference plane. The external forces on the vehicle are the tire forces F_1 and F_2 on the rear and front pairs of wheels, respectively, and the force F_p of the passenger on its seat, which is a distance x_p forward of the rear tire contact point. The center of gravity of the empty vehicle is at a distance x_1 forward of the rear-tire contact point, and at a distance x_2 aft of the front-tire contact point, so the wheelbase L is $x_1 + x_2$. Let $x_{1p} \equiv x_1 - x_p$. The horizontal drag forces are balanced by the thrust of the motor. The internal paper "Deflection of the Running Surface" shows that with the current design, a midspan deflection between supports 54 inches apart of 0.023 inch can be expected with half-inch thick angles. The question here is: "Is this adequate?" The conclusion of this paper is that ride comfort is more than acceptable if the midspan deflection is 0.055". Thus we can use the half-inch angle.

3. Deflections

Let the running surface be described by

$$z_s = z_o e^{2\pi i x_{cg}/\lambda} = z_o e^{i\omega t} \tag{2}$$

where λ is the wave length of a sinusoidally varying running surface and $2z_o$ is the deflection from peak to trough. In equation (2)

$$\omega = 2\pi\frac{V}{\lambda} \tag{3}$$

and

$$f = \frac{V}{\lambda} \tag{4}$$

is called the "crossing frequency."

When the vehicle is moving along with its c. g. bouncing up and down an amount $z(t)$ and rocking back and forth at an angle $\theta(t)$ the deflections of the rear and front tires, respectively, using equation (2) are given by the equations

$$\begin{gathered}\delta_1 = -z + x_1\theta + z_o e^{2\pi i(x_{cg}-x_1)/\lambda} = -z + x_1\theta + z_o e^{-2\pi i x_1} e^{i\omega t} \\ \dot{\delta}_1 = -\dot{z} + x_1\dot{\theta} + i\omega z_o e^{-2\pi i x_1} e^{i\omega t}\end{gathered} \tag{5}$$

$$\begin{gathered}\delta_2 = -z - x_2\theta + z_o e^{2\pi i(x_{cg}+x_2)/\lambda} = -z - x_2\theta + z_o e^{2\pi i x_2} e^{i\omega t} \\ \dot{\delta}_2 = -\dot{z} - x_2\dot{\theta} + i\omega z_o e^{2\pi i x_2} e^{i\omega t}\end{gathered} \tag{6}$$

The vertical position of the unloaded passenger seat is at

$$z_{seat} = z_{seat_0} + \Delta z - x_{1p}\theta, \qquad \dot{z}_{seat} = \dot{\Delta z} - x_{1p}\dot{\theta} \tag{7}$$

The vertical position of the passenger, represented as a point, is at $z_p < z_{seat}$, which means that the spring force of the seat is proportional to $z_{seat} - z_p$.

The static balance of forces is obtained by taking the static moment about both the front and rear tire contact points. Thus

$$\begin{gathered}F_1 L = 2k\delta_1 L = Wx_2 + W_p\left(x_{1p} + x_2\right) \\ F_2 L = 2k\delta_2 L = Wx_1 + W_p x_p \\ 2k(\delta_1 + \delta_2) = W + W_p \\ \delta_m = \frac{\delta_1 + \delta_2}{2} = \frac{W_{gross}}{4k}\end{gathered} \tag{8}$$

4. The Forces and Equation of Motion

The force-deflection relationship of each of the pair of front and back tires is linear, each with a spring constant k and damping constant c. Thus, the forces on the rear and front pair of tires are given respectively by

$$F_1 = 2k\delta_1 + 2c\dot{\delta}_1 \equiv 2k(\delta_{static1} + \Delta\delta_1) + 2c\dot{\Delta\delta}_1$$

(9)

$$F_2 = 2k\delta_2 + 2c\dot{\delta}_2 \equiv 2k(\delta_{static2} + \Delta\delta_2) + 2c\Delta\dot{\delta}_2$$

(10)

The downward force of the passenger on the seat is

$$F_p = k_s(z_{seat} - z_p) + c_s(\dot{z}_{seat} - \dot{z}_p)$$

(11)

The equations of motion of the vehicle are

$$\frac{W}{g}\ddot{z} = -W + F_1 + F_2 - F_p = 2k(\Delta\delta_1 + \Delta\delta_2) + 2c(\Delta\dot{\delta}_1 + \Delta\dot{\delta}_2) - F_p$$

(12)

$$\frac{W}{g}r_g^2\ddot{\theta} = F_2x_2 - F_1x_1 + F_px_{1p} = 2k(\Delta\delta_2x_2 - \Delta\delta_1x_1) + 2c(\Delta\dot{\delta}_2x_2 - \Delta\dot{\delta}_1x_1) + F_px_{1p}$$

(13)

in which r_g is the radius of gyration of the vehicle and it is assumed that the static tire forces equal the weight of the vehicle and passenger. The equation of motion of the passenger is

$$\begin{aligned}\frac{W_p}{g}\ddot{z}_p &= -W_p + k_s(z_{seat} - z_p) + c_s(\dot{z}_{seat} - \dot{z}_p)\\ &= k_s(\Delta z - x_{1p}\theta - z_p) + c_s(\Delta\dot{z} - x_{1p}\dot{\theta} - \dot{z}_p)\end{aligned}$$

(14)

in which it is assumed that the static force of the seat on the passenger is balanced by the passenger weight.

4. The Solution

The solution of equations (12), (13) and (14) for oscillatory motion can be expressed as

$$\Delta z = z_m e^{i\omega t}, \qquad \theta = \theta_m e^{i\omega t}, \qquad z_p = z_{pm}e^{i\omega t}$$

(15)

where the subscript *m* indicates the maximum value measured from the mean position. Substitute the solutions (15) and the variable part of the deflections (5), (6), and (8) into the equations of motion (12), (13) and (14) and define

$$\omega_n^2 = \frac{4kg}{W}, \;\; 2\zeta\omega_n = \frac{cg}{W}$$

$$\omega_{sv}^2 = \frac{k_sg}{W}, \qquad 2\zeta_s\omega_{sv} = \frac{c_sg}{W}, \qquad \omega_{seat}^2 = \frac{k_sg}{W_p}, \qquad 2\zeta_s\omega_{seat} = \frac{c_sg}{W_p}$$

(17)

After dividing by $e^{i\omega t}$ and *W/g*, equations (12) and (13) become

$$-\omega^2 z_m = -\left(\frac{\omega_n^2}{2} + 4\varsigma\omega_n i\omega\right)\left[2z_m + (x_2 - x_1)\theta_m - z_o\left(e^{i2\pi\frac{x_2}{\lambda}} + e^{-i2\pi\frac{x_1}{\lambda}}\right)\right]$$
$$- (\omega_{sv}^2 + 2i\omega\zeta_s\omega_{sv})\big(z_m - x_{1p}\theta_m - z_{pm}\big) \tag{18}$$

$$-\omega^2 r_g^2\theta_m = -\left(\frac{\omega_n^2}{2} + 4\zeta\omega_n i\omega\right)\left[z_m(x_2 - x_1) + \theta_m(x_2^2 + x_1^2) - z_o\left(x_2 e^{2\pi i\frac{x_2}{\lambda}} - x_1 e^{-2\pi i\frac{x_1}{\lambda}}\right)\right]$$
$$+ (\omega_{sv}^2 + 2i\omega\zeta_s\omega_{sv})\big(z_m - x_{1p}\theta_m - z_{pm}\big)x_{1p} \tag{19}$$

After dividing by W_p/g and $e^{i\omega}$ equation (14) becomes

$$-\omega^2 z_{pm} = (\omega_{seat}^2 + 2i\omega\zeta_s\omega_{seat})\big(z_m - x_{1p}\theta_m - z_{pm}\big) \tag{20}$$

Let

$$P \equiv \omega_n^2\left(\frac{x_1^2 + x_2^2}{2}\right) - \omega^2 r_g^2 + \omega_{sv}^2 x_{1p}^2 \tag{21}$$

$$Q \equiv \omega_n^2\left(\frac{x_2 - x_1}{2}\right) - \omega_{sv}^2 x_{1p} \tag{22}$$

$$R \equiv 4\zeta\omega_n\omega(x_2 - x_1) - 2\zeta_s\omega_{sv}\omega x_{1p} \tag{23}$$

$$S \equiv 4\zeta\omega_n\omega(x_1^2 + x_2^2) + 2\zeta_s\omega_{sv}\omega x_{1p}^2 \tag{24}$$

Then, equations (18), (19) and (20) become

$$(\omega_n^2 - \omega^2 + 8i\omega\zeta\omega_n + \omega_{sv}^2 + 2i\omega\zeta_s\omega_{sv})z_m + (Q + iR)\theta_m - (\omega_{sv}^2 + 2i\omega\zeta_s\omega_{sv})z_{pm}$$
$$= \left(\frac{\omega_n^2}{2} + 4i\omega\zeta\omega_n\right)z_o\left(e^{i2\pi\frac{x_2}{\lambda}} + e^{-i2\pi\frac{x_1}{\lambda}}\right) \tag{25}$$

$$(Q + iR)z_m + (P + iS)\theta_m + (\omega_{sv}^2 + 2i\omega\zeta_s\omega_{sv})x_{1p}z_{pm}$$
$$= \left(\frac{\omega_n^2}{2} + 4i\omega\zeta\omega_n\right)z_o\left(x_2 e^{i2\pi\frac{x_2}{\lambda}} - x_1 e^{-i2\pi\frac{x_1}{\lambda}}\right) \tag{26}$$

$$\omega_{seat}(\omega_{seat} + 2i\omega\zeta_s)\big(z_m - x_{1p}\theta_m\big) + [\omega^2 - \omega_{seat}(\omega_{seat} + 2i\omega\zeta_s)]z_{pm} = 0 \tag{27}$$

We are interested in the amplitude of the vertical acceleration of the passenger,

$$\left|\ddot{z}_{pm}\right| = \omega_{seat}^2\left|z_{pm}\right| \tag{28}$$

We must solve equations (25), (26) and (27) for z_{pm}. To do so without excessive algebra, let these three equations be represented as follows:

$$
\begin{aligned}
A\, z_m + B\theta_m + C z_{pm} &= R_1 \\
D z_m + E\theta_m + F z_{pm} &= R_2 \\
G z_m + H\theta_m + K z_{pm} &= 0
\end{aligned}
\tag{29}
$$

in which

$$
\begin{aligned}
&A = \omega_n^2 - \omega^2 + \omega_{sv}^2 + 2i\omega(4\zeta\omega_n + \zeta_s\omega_{sv}) \\
&B = Q + iR \\
&C = -\omega_{sv}(\omega_{sv} + 2i\omega\zeta_s) \\
&D = B \\
&E = P + iS \\
&F = -Cx_{1p} \\
&G = \omega_{seat}(\omega_{seat} + 2i\omega\zeta_s) \\
&H = -Gx_{1p} \\
&K = \omega^2 - G \\
&R_1 = z_o\omega_n\left(\frac{\omega_n}{2} + 4i\omega\zeta\right)\left(e^{i2\pi\frac{x_2}{\lambda}} + e^{-i2\pi\frac{x_1}{\lambda}}\right) \\
&R_2 = z_o\omega_n\left(\frac{\omega_n}{2} + 4i\omega\zeta\right)\left(x_2 e^{i2\pi\frac{x_2}{\lambda}} - x_1 e^{-i2\pi\frac{x_1}{\lambda}}\right)
\end{aligned}
$$

The third of equations (29) can be written as

$$
G z_m = -H\theta_m - K z_{pm}
$$

Multiply the first two of equations by G and substitute for $G z_m$ to get

$$
\begin{aligned}
-A\left(H\theta_m + K z_{pm}\right) + GB\theta_m + GC z_{pm} &= GR_1 \\
-D\left(H\theta_m + K z_{pm}\right) + GE\theta_m + GF z_{pm} &= GR_2
\end{aligned}
$$

or

$$
\begin{aligned}
(GB - AH)\theta_m + (GC - AK) z_{pm} &= GR_1 \\
(GE - DH)\theta_m + (GF - DK) z_{pm} &= GR_2
\end{aligned}
$$

Cross multiply and subtract:

$$
\begin{aligned}
(GE - DH)\big[(GB - AH)\theta_m + (GC - AK) z_{pm} &= GR_1\big] \\
(GB - AH)\big[(GE - DH)\theta_m + (GF - DK) z_{pm} &= GR_2\big]
\end{aligned}
$$

$$
[(GE - DH)(GC - AK) - (GB - AH)(GF - DK)] z_{pm} = G[(GE - DH)R_1 - (GB - AH)R_2]
$$

or

$$z_{pm} = G\frac{(GE-DH)R_1-(GB-AH)R_2}{(GE-DH)(GC-AK)-(GB-AH)(GF-DK)}$$

(30)

Each of the terms in equation (30) has a real part and an imaginary part. Use the subscript r for the real part and i for the imaginary part, and take into account the trigonometric identity $e^{\pm ia} \equiv \cos a \pm i \sin a$. The terms in equation (30), with the coefficient on the right side shown first, are

$$\Psi_r + i\Psi_i \equiv \omega_n\omega_{seat}\left(\frac{\omega_n}{2}+4i\omega\zeta\right)(\omega_{seat}+2i\omega\zeta_s)$$
$$= \omega_n\omega_{seat}\left[\frac{\omega_n\omega_{seat}}{2}-8\omega^2\zeta\zeta_s+i\omega(4\zeta\omega_{seat}+\zeta_s\omega_n)\right]$$

$$(GE-DH) = (G_r+iG_i)(E_r+iE_i)-(D_r+iD_i)(H_r+iH_i) =$$
$$(G_rE_r-G_iE_i-D_rH_r+D_iH_i)+i(G_iE_r+G_rE_i-D_rH_i-D_iH_r)$$
$$\equiv GEDH_r+iGEDH_i$$
$$(GB-AH) = (G_rB_r-G_iB_i-A_rH_r+A_iH_i)+i(G_iB_r+G_rB_i-A_rH_i-A_iH_r)$$
$$\equiv GBAH_r+iGBAH_i$$
$$(GC-AK) = (G_rC_r-G_iC_i-A_rK_r+A_iK_i)+i(G_iC_r+G_rC_i-A_rK_i-A_iK_r)$$
$$\equiv GCAK_r+iGCAK_i$$
$$(GF-DK) = (G_rF_r-G_iF_i-D_rK_r+D_iK_i)+i(G_iF_r+G_rF_i-D_rK_i-D_iK_r)$$
$$\equiv GFDK_r+iGFDK_i$$
$$e^{i2\pi\frac{x_2}{\lambda}}+e^{-i2\pi\frac{x_1}{\lambda}} = cos2\pi\frac{x_2}{\lambda}+cos2\pi\frac{x_1}{\lambda}+i\left(sin2\pi\frac{x_2}{\lambda}-sin2\pi\frac{x_1}{\lambda}\right)\equiv\alpha_r+i\alpha_i$$
$$x_2e^{i2\pi\frac{x_2}{\lambda}}-x_1e^{-i2\pi\frac{x_1}{\lambda}} = x_2cos2\pi\frac{x_2}{\lambda}-x_1cos2\pi\frac{x_1}{\lambda}+i\left(x_2sin2\pi\frac{x_2}{\lambda}+x_1sin2\pi\frac{x_1}{\lambda}\right)\equiv\beta_r+i\beta_i$$

Thus, equation (30) can be written in the form

$$z_{pm} = z_o(\Psi_r+i\Psi_i)$$
$$\times\frac{[(GEDH_r+iGEDH_i)(\alpha_r+i\alpha_i)-(GBAH_r+iGBAH_i)(\beta_r+i\beta_i)]}{(GEDH_r+iGEDH_i)(GCAK_r+iGCAK_i)-(GBAH_r+iGBAH_i)(GFDK_r+iGFDK_i)}$$
$$Num_r = GEDH_r\alpha_r-GEDH_i\alpha_i-GBAH_r\beta_r+GBAH_i\beta_i$$
$$Num_i = GEDH_r\alpha_i+GEDH_i\alpha_r-GBAH_r\beta_i-GBAH_i\beta_r$$
$$Den_r = GEDH_r\,GCAK_r-GEDH_i\,GCAK_i-GBAH_r\,GFDK_r+GBAH_i\,GFDK_i$$
$$Den_i = GEDH_r\,GCAK_i+GEDH_i\,GCAK_r-GBAH_r\,GFDK_i-GBAH_i\,GFDK_r$$
$$z_{pm} = z_o(\Psi_r+i\Psi_i)\frac{Num_r+iNum_i}{Den_r+iDen_i}\times\frac{Den_r-iDen_i}{Den_r-iDen_i}$$

$$= z_o(\Psi_r+i\Psi_i)\frac{Num_rDen_r+Num_iDen_i+i(Num_iDen_r-Num_rDen_i)}{Den_r^2+Den_i^2}$$
$$= z_o\frac{\Psi_rNumDen_{rrii}-\Psi_iNumDen_{irri}+i[\Psi_rNumDen_{irri}+\Psi_iNumDen_{rrii}]}{Den_r^2+Den_i^2}$$

(31)

The ISO ride-comfort standard is given in terms of the root-mean-square (r.m.s.) acceleration, which is the peak acceleration divided by $\sqrt{2}$. Thus the absolute value of the r.m.s. acceleration of the passenger c.g. in g's can be expressed as

$$\frac{|\ddot{z}_p|_{rms}}{g} = \frac{\omega_{seat}^2 z_o}{g\sqrt{2}} \left(\frac{\sqrt{N_r^2 + N_i^2}}{\mathrm{Den}_\mathrm{r}^2 + \mathrm{Den}_\mathrm{i}^2} \right) \tag{32}$$

Equation (32), with the notation defined is programmed for numerical solution in the Appendix. Since $\ddot{z}_p$ is proportional to the amplitude of the sinusoidal variation in the running surface, z_o , we calculate equation (32) for one value and the reader can easily determine $\ddot{z}_p$ for other values.

5. Discussion

Results obtained by running the program given in the Appendix are tabulated below for twelve of the thirteen parameters listed first, and results are given for a range of speeds. The first set of results is for a 'best' set of parameter. Note that the calculated passenger acceleration is far below the comfort limits, which are computed via a function listed at the end of the program, and based on ISO standards for a 25-minute ride.

For the second set of results, I have increased the seat stiffness by a factor of 10. In this case ride comfort exceeds the ISO standard, showing that seat suspension is essential to adequate ride comfort.

For the third set of results, seat stiffness is restored to the previous value, but the weight of the passenger is reduced to 50 lb. In this case the ride-comfort standard is exceeded only at the lowest speed. By reducing the seat stiffness to 400 lb/in, passenger acceleration reduces to 0.119 g, which is below the standard.

For the fourth set of results, the vehicle c. g. and the passenger are both moved forward to the midpoint between the front and rear tires. At the lowest speeds, passenger acceleration is increased slightly, but it is still much below the standard with a 200 lb passenger.

Any other combination of parameters can be tried, but are left to the systems engineering team.

PARAMETERS USED TO DETERMINE RIDE COMFORT

Units are lb, in

Vehicle empty weight	1200
Passenger weight	200
Vehicle radius of gyration	40
Tire stiffness	1500
Tire damping ratio	0.3

Seat stiffness	800
Seat damping ratio	0.3
Distance between running surface supports	54
Distance between front and rear tires	80
Distance of vehicle c.g. from rear tires	24
Distance of passenger forward of rear tires .	16
Midspan deflection of running surface . . .	0.055

Vehicle Speed	Frequency	Passenger Acceleration	Comfort Limit
mph	Hz	g's	g's
10	3.259	0.027	0.214
15	4.889	0.009	0.180
20	6.519	0.005	0.180
25	8.148	0.003	0.182
30	9.778	0.002	0.210
35	11.407	0.001	0.242
40	13.037	0.001	0.278
45	14.667	0.001	0.321
50	16.296	0.001	0.369
55	17.926	0.001	0.425
60	19.556	0.000	0.490

PARAMETERS USED TO DETERMINE RIDE COMFORT
Units are lb, in

Vehicle empty weight	1200
Passenger weight	200
Vehicle radius of gyration	40
Tire stiffness	1500
Tire damping ratio	0.3
Seat stiffness	8000
Seat damping ratio	0.3
Distance between running surface supports	54
Distance between front and rear tires . .	80
Distance of vehicle c.g. from rear tires	24
Distance of passenger forward of rear tires	16
Midspan deflection of running surface . .	0.055

Vehicle Speed	Frequency	Passenger Acceleration	Comfort Limit
mph	Hz	g's	g's
10	3.259	0.672	0.214
15	4.889	0.542	0.180
20	6.519	0.302	0.180

25	8.148	0.156	0.182
30	9.778	0.091	0.210
35	11.407	0.059	0.242
40	13.037	0.042	0.278
45	14.667	0.031	0.321
50	16.296	0.024	0.369
55	17.926	0.019	0.425
60	19.556	0.016	0.490

PARAMETERS USED TO DETERMINE RIDE COMFORT
Units are lb, in

Vehicle empty weight 1200
Passenger weight 50
Vehicle radius of gyration 40
Tire stiffness 1500
Tire damping ratio 0.3
Seat stiffness 800
Seat damping ratio 0.3
Distance between running surface supports 54
Distance between front and rear tires . . 80
Distance of vehicle c.g. from rear tires 24
Distance of passenger forward of rear tires 16
Midspan deflection of running surface . . 0.055

Vehicle Speed mph	Frequency Hz	Passenger Acceleration g's	Comfort Limit g's
10	3.259	0.386	0.214
15	4.889	0.136	0.180
20	6.519	0.054	0.180
25	8.148	0.029	0.182
30	9.778	0.018	0.210
35	11.407	0.012	0.242
40	13.037	0.009	0.278
45	14.667	0.007	0.321
50	16.296	0.005	0.369
55	17.926	0.004	0.425
60	19.556	0.004	0.490

PARAMETERS USED TO DETERMINE RIDE COMFORT
Units are lb, in

Vehicle empty weight 1200
Passenger weight 200
Vehicle radius of gyration 40
Tire stiffness 1500
Tire damping ratio 0.3
Seat stiffness 800
Seat damping ratio 0.3
Distance between running surface supports 54
Distance between front and rear tires . . 80
Distance of vehicle c.g. from rear tires 40
Distance of passenger forward of rear tires 40
Midspan deflection of running surface . . 0.055

Vehicle Speed	Frequency	Passenger Acceleration	Comfort Limit
mph	Hz	g's	g's
10	3.259	0.025	0.214
15	4.889	0.012	0.180
20	6.519	0.007	0.180
25	8.148	0.005	0.182
30	9.778	0.004	0.210
35	11.407	0.003	0.242
40	13.037	0.002	0.278
45	14.667	0.002	0.321
50	16.296	0.001	0.369
55	17.926	0.001	0.425
60	19.556	0.001	0.490

Appendix

```
'This program "RIDECOMF.BAS" calculates the vertical accelera-
tion of a
'passenger riding in an ITNS vehicle on a flexible guideway.
'Units are inch, pound, seconds

DEFDBL A-Z
DEFINT I
DECLARE FUNCTION Acomfort (f)
g = 32.174        'ft/s^2
Pi = 4 * ATN(1)
CLS
W = 1200          'empty weight of vehicle, lb
```

```
Wp = 200        'weight of passenger, lb
rg = 40         'in, radius of gyration of vehicle
k = 1500        'tire stiffness, lb/in
Zeta.tire = .3  'tire damping ratio
k.seat = 800    'seat stiffness, lb/in
Zeta.seat = .3  'seat damping ratio
Lambda = 54     'distance between supports, in
Wheelbase = 80  'in
Veh.cg = .5 * Wheelbase 'Position of vehicle c.g. forward of
rear tire
xp = .5 * Wheelbase     'Position of passenger forward of rear
tire
MidspanDeflection = .055
zo = MidspanDeflection / 2

PRINT "PARAMETERS USED TO DETERMINE RIDE COMFORT"
PRINT "         Units are lb, in"
PRINT
PRINT "Vehicle empty weight . . . . . . . . . . .  ";
PRINT USING "####"; W
PRINT "Passenger weight . . . . . . . . . . . . .  ";
PRINT USING "####"; Wp
PRINT "Vehicle radius of gyration . . . . . . . .  ";
PRINT USING "####"; rg
PRINT "Tire stiffness . . . . . . . . . . . . . .  ";
PRINT USING "####"; k
PRINT "Tire damping ratio . . . . . . . . . . . .  ";
PRINT USING "####.#"; Zeta.tire
PRINT "Seat stiffness . . . . . . . . . . . . . .  ";
PRINT USING "####"; k.seat
PRINT "Seat damping ratio . . . . . . . . . . . .  ";
PRINT USING "####.#"; Zeta.seat
PRINT "Distance between running surface supports   ";
PRINT USING "####"; Lambda
PRINT "Distance between front and rear tires  . .  ";
PRINT USING "####"; Wheelbase
PRINT "Distance of vehicle c.g. from rear tires    ";
PRINT USING "####"; Veh.cg
PRINT "Distance of passenger forward of rear tires ";
PRINT USING "####"; xp
PRINT "Midspan deflection of running surface  . .  ";
PRINT USING "####.###"; MidspanDeflection
PRINT

Omega.n = SQR(4 * k * g / W)
Omega.seat = SQR(k.seat * g / Wp)
Omega.sv = SQR(k.seat * g / W)
```

```
x1 = Vehcg               'distance from rear tire to vehicle c.g.
x2 = Wheelbase - x1      'distance from front tire to vehicle
c.g.
x1p = x1 - xp            'distance of pasenger behind vehicle
c.g.

PRINT "  Vehicle Speed  Frequency  Passenger Acceleration  Com-
fort Limit"
PRINT "     mph             Hz              g's
g's"

FOR I = 10 TO 60 STEP 5 'vehicle speed in mph
V = I * 88 / 60          'vehicle speed, ft/sec
Omega = 2 * Pi * V / (Lambda / 12)      'Radial forcing fre-
quency
Frequency = Omega / 2 / Pi
Term = x1 ^ 2 + x2 ^ 2
P = .5 * Omega.n ^ 2 * Term - Omega ^ 2 * rg ^ 2 + Omega.sv ^ 2
* x1p ^ 2
Q = .5 * Omega.n ^ 2 * (x2 - x1) - Omega.sv ^ 2 * x1p
R = 2 * Omega * (2 * Zeta.tire * Omega.n * (x2 - x1) - Zeta.seat
* Omega.sv * x1p)
S = 2 * Omega * (2 * Zeta.tire * Omage.n * Term + Zeta.seat *
Omega.sv * x1p ^ 2)

Ar = Omega.n ^ 2 - Omega ^ 2 + Omega.sv ^ 2
Ai = 2 * Omega * (4 * Zeta.tire * Omega.n + Zeta.seat *
Omega.sv)
Br = Q
Bi = R
Cr = -Omega.sv ^ 2
Ci = -2 * Omega.sv * Omega * Zeta.seat
Dr = Br
Di = Bi
Er = P
Ei = S
Fr = -Cr * x1p
Fi = -Ci * x1p
Gr = Omega.seat ^ 2
Gi = 2 * Omega * Omega.seat * Zeta.seat
Hr = -Gr * x1p
Hi = -Gi * x1p
Kr = Omega ^ 2 - Gr
Ki = -Gi

Term = Omega.n * Omega.seat
```

```
Psi.r = Term * (Term / 2 - 8 * Omega ^ 2 * Zeta.tire *
Zeta.seat)
Psi.i = Term * Omega * (4 * Zeta.tire * Omega.seat + Zeta.seat *
Omega.n)

GEDHr = Gr * Er - Gi * Ei - Dr * Hr + Di * Hi
GEDHi = Gi * Er + Gr * Ei - Dr * Hi - Di * Hr

GBAHr = Gr * Br - Gi * Bi - Ar * Hr + Ai * Hi
GBAHi = Gi * Br + Gr * Bi - Ar * Hi - Ai * Hr

GCAKr = Gr * Cr - Gi * Ci - Ar * Kr + Ai * Ki
GCAKi = Gi * Cr + Gr * Ci - Ar * Ki - Ai * Kr

GFDKr = Gr * Fr - Gi * Fi - Dr * Kr + Di * Ki
GFDKi = Gi * Fr + Gi * Fr - Dr * Ki - Di * Kr

Alpha.r = COS(2 * Pi * x2 / Lambda) + COS(2 * Pi * x1 / Lambda)
Alpha.i = SIN(2 * Pi * x2 / Lambda) - SIN(2 * Pi * x1 / Lambda)
Beta.r = x2 * COS(2 * Pi * x2 / Lambda) - x1 * COS(2 * Pi * x1 /
Lambda)
Beta.i = x2 * SIN(2 * Pi * x2 / Lambda) + x1 * SIN(2 * Pi * x1 /
Lambda)

Num.r = GEDHr * Alpha.r - GEDHi * Alpha.i - GBAHr * Beta.r +
GBAHi * Beta.i
Num.i = GEDHr * Alpha.i + GEDHi * Alpha.r - GBAHr * Beta.i -
GBAHi * Beta.r
Den.r = GEDHr * GCAKr - GEDHi * GCAKi - GBAHr * GFDKr + GBAHi *
GFDKi
Den.i = GEDHr * GCAKi + GEDHi * GCAKr - GBAHr * GFDKi - GBAHi *
GFDKr

NumDen.rrii = Num.r * Den.r + Num.i * Den.i
NumDen.irri = Num.i * Den.r - Num.r * Den.i

Nr = Psi.r * NumDen.rrii - Psi.i * NumDen.irri
Ni = Psi.r * NumDen.irri + Psi.i * NumDen.rrii

Coeff = Omega.seat ^ 2 * zo / g / SQR(2)
RMSPassAccel = Coeff * SQR(Nr ^ 2 + Ni ^ 2) / (Den.r ^ 2 + Den.i
^ 2)

PRINT "     "; I;
PRINT USING "##########.###"; Frequency;
PRINT USING "#############.###"; RMSPassAccel;
ComfortAccel = Acomfort(Frequency)
```

```
PRINT USING "################.###"; ComfortAccel
NEXT I

FUNCTION Acomfort (f)
  IF f < 1 THEN
     Acomfort = .36
  ELSEIF f < 4 THEN
     ex = (f - 1) / 3
     Acomfort = .36 / 2 ^ ex
  ELSEIF f < 8 THEN
     Acomfort = .18
  ELSE
     ex = (f - 8) / 8
     Acomfort = .18 * 2 ^ ex
  END IF
END FUNCTION
```

Deflection of the Running Surface

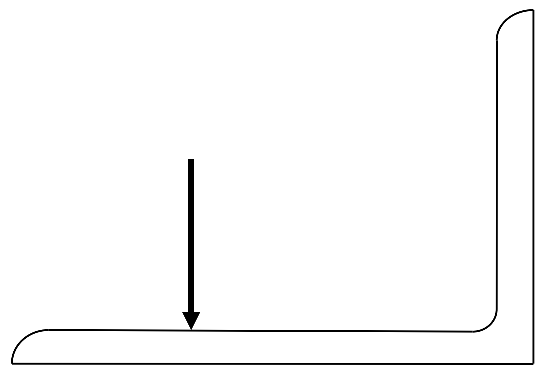


Figure 1. Load on Angle Running Surface

The main support wheels of the ***ITNS*** vehicle run on a pair of steel angles supported a distance *L* apart. The problem we address here relates to the need to move the main-support tires inward enough to allow space for a communication (leaky) cable between the vertical side of the angle and the outer side of the tire, and to determine the angle thickness needed to maintain the vertical acceleration below the ISO Comfort Standard. Moving the tires inward increases the deflection due to twist of the angle. The purpose of this paper is to determine the angle thickness and position of the tire load *P* that will keep the vertical acceleration below the ISO comfort limit.

<u>The Deflection of the Angle under the Wheel Load</u>

The deflection is due to a combination of bending and torsion. From the paper "Deflection of a Continuous Beam resting on Regularly-Spaced Simple Supports under a Concentrated Load" the deflection at the midpoint between simple supports holding up a continuous beam is

$$\Delta = 0.2573\,\frac{PL^3}{48EI}$$

in which the modulus of elasticity for steel is $E = 29.5(10)^6$ psi. I is the moment of inertia of the angle.

From page 270 of Timoshenko *Strength of Materials: Part II* the twist per unit of length of a large aspect-ratio cross section is

$$\theta = \frac{M_t}{\frac{1}{3}bc^3G}$$

in which b is the sum of the lengths of the two legs ($b = 8+6 = 14''$), c is the angle thickness, and G is the shear modulus, $11.5(10)^6$ psi for steel. $\frac{E}{G} = 2(1+\mu)$, where μ is the Poisson's Ratio. If a is the distance between the applied load P and the outer edge of the angle and x_p is the distance between the neutral axis of the angle and the outer edge of the angle, the torsional moment is

$$M_t = P(a - x_p)$$

The angle of twist per unit length due to torsion is then

$$\theta = \frac{3P(a - x_p)}{bc^3G}$$

The deflection of the angle under the load P due to torsion is

$$\Delta_{torsion} = \theta\frac{L}{2}(a - x_p) = \frac{3PL(a - x_p)^2}{2bc^3G}$$

Thus, the total deflection is

$$\Delta_{total} = \frac{PL}{2E}\left[\frac{L^2}{93.28I} + \frac{6(1+\mu)(a - x_p)^2}{bc^3}\right]$$

in which for steel $\mu = 0.283$. The maximum load on one wheel will be about 500 lb, and the distance between supports is L = 54 inches.

From the *Manual of Steel Construction*, with the 6 in leg of the angle vertical we find

Angle thickness, c, in	I, in^4	x_p, in
0.5000	21.7	0.425
0.5625	24.1	0.476

For the half-inch thick angle

$$\Delta_{total} = \frac{500(54)}{2(29.5)(10)^6}\left[\frac{54^2}{93.28(21.7)} + \frac{6(1.283)(3.75 - .425)^2}{14(0.5)^3}\right]$$
$$= 0.0004576(1.441 + 48.63) = 0.0229''$$

Deflection of a Continuous Beam resting on Regularly-Spaced Simple Supports under a Concentrated Load

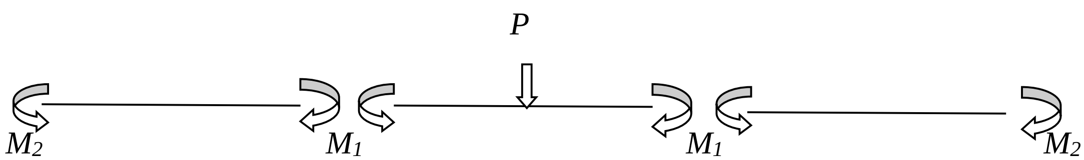


Figure 1. Sections of a Continuous Beam

Consider a continuous beam having many sections each of length *l*, three of which are shown in Figure 1. The beam is supported on simple supports at each break point. One point load *P* is applied on the middle span, at the left and right of which moment M_1 is applied. Thus the configuration is symmetric about the load *P*. The beams are connected by the condition that the slopes must be continuous.

1. The Loaded Beam

From structural mechanics, the structural rigidity *EI*, where *E* is the modulus of elasticity and *I* is the area moment of inertia of the beam, multiplied by the curvature of the beam is equal to the applied moment. For small deflections, the curvature can be taken as the second derivative of the deflection. Let the deflection be *y* and the coordinate in the direction of the beam be *x* with *x* = 0 at the left support. Thus, for the loaded beam, where the reactions at the left and right supports are each *P*/2,

$$EIy'' = M_1 - \frac{1}{2}Px + P\langle x - l/2\rangle \tag{1}$$

in which $\langle x - \frac{l}{2}\rangle = x - \frac{l}{2}\ if\ x - \frac{l}{2} > 0$ and $\langle x - \frac{l}{2}\rangle = 0$ if $x - l/2 \leq 0$. Integrating once

$$EIy' = M_1 x - \frac{Px^2}{4} + P\frac{\langle x - l/2\rangle^2}{2} + C \tag{2}$$

in which *C* is a constant. Integrating again

$$EIy = M_1\frac{x^2}{2} - \frac{Px^3}{12} + P\frac{\langle x - l/2\rangle^3}{6} + Cx \tag{3}$$

which meets the condition that the deflection *y* is zero at *x* = 0. The deflection is also zero at *x* = l. Thus

$$EIy(l) = 0 = l\left[M_1\frac{l}{2} - \frac{Pl^2}{12} + \frac{Pl^2}{48} + C\right] \tag{4}$$

The mid-span deflection $y(l/2) \equiv \delta$ is therefore

$$EI\delta = M_1 \frac{l^2}{8} - \frac{Pl^3}{96} - \left(M_1 \frac{l}{2} - \frac{Pl^2}{12} + \frac{Pl^2}{48}\right)\frac{l}{2}$$

or

$$\delta = \frac{Pl^3}{48EI} - \frac{M_1 l^2}{8EI} \tag{5}$$

Thus, with $M_1 = 0$, we get the well-known formula for the deflection of a simple beam under a point load. The slope at the left end of this beam is

$$EIy'(0) = C = \frac{Pl^2}{16} - M_1 \frac{l}{2} \tag{6}$$

2. <u>The First Unloaded Beams</u>

Now consider the first unloaded span to the left of the loaded span. Applying the same structural theory we get

$$EIy'' = M_2 + \left(\frac{M_1 - M_2}{l}\right)x \tag{7}$$

in which $\left(\frac{M_1 - M_2}{l}\right)$ is the downward reaction at the left end of this beam. Integrating once

$$EIy' = M_2 x + \left(\frac{M_1 - M_2}{l}\right)\frac{x^2}{2} + C \tag{8}$$

Integrating again

$$EIy = M_2 \frac{x^2}{2} + \left(\frac{M_1 - M_2}{l}\right)\frac{x^3}{6} + Cx \tag{9}$$

Since $y = 0$ at both ends of this beam, we have for the right end

$$M_2 \frac{l}{2} + \left(\frac{M_1 - M_2}{l}\right)\frac{l^2}{6} + C = 0$$

or

$$C = -\frac{1}{3} M_2 l - \frac{1}{6} M_1 l \tag{10}$$

Thus,

$$EIy' = M_2\left(x - \frac{x^2}{2l} - \frac{l}{3}\right) + M_1\left(\frac{x^2}{2l} - \frac{l}{6}\right)$$

Thus

$$EIy'(0) = -\frac{1}{3}M_2 l - \frac{1}{6}M_1 l \text{ and } EIy'(l) = \frac{1}{6}M_2 l + \frac{1}{3}M_1 l \tag{11}$$

The slope at the right end of this beam must be equal to the slope at the left end of the loaded beam. Thus, using equations (11) and (6)

$$\frac{1}{3}M_2 l + \frac{1}{6}M_1 l = \frac{Pl^2}{16} - M_1\frac{l}{2}$$

or

$$\frac{1}{3}(2M_1 + M_2) = \frac{Pl}{16}$$

or

$$M_1 = -\frac{1}{2}M_2 + \frac{3}{32}Pl \tag{12}$$

Then, substitute M_1 from equation (12) into equation (5) to get

$$\delta = \frac{Pl^3}{48EI} - \frac{M_1 l^2}{8EI} = \frac{Pl^3}{48EI} + \frac{M_2 l^2}{16EI} - \frac{9Pl^3}{48 \times 16EI} = \frac{7}{16} \times \frac{Pl^3}{48EI} + \frac{M_2 l^2}{16EI} \tag{13}$$

If $M_2 = 0$, i.e., if there is only one unloaded beam on each side of the loaded beam, the deflection is reduced to 7/16th of the value with only the single beam.

3. <u>Second Unloaded Beams</u>

If there is a next beam on each side of the middle three, i.e., a total of five beams, the slope at the left end of the three beams must equal the slope at the right end of the right hand beam. From equation (11), for this second beam

$$EIy'(0) = -\frac{1}{3}M_3 l - \frac{1}{6}M_2 l \text{ and } EIy'(l) = \frac{1}{6}M_3 l + \frac{1}{3}M_2 l \tag{14}$$

Thus, at the position of M_2 we have

$$-\frac{1}{3}M_2 l - \frac{1}{6}M_1 l = \frac{1}{6}M_3 l + \frac{1}{3}M_2 l$$

or

$$M_1 + 4M_2 + M_3 = 0 \tag{15}$$

This is the well-known theorem of three moments.

Substituting for M_1 from equation (12), we get

$$-\frac{1}{2}M_2 + \frac{3}{32}Pl + 4M_2 + M_3 = 0$$

or

$$M_2 = -\frac{3}{7}\left(\frac{3}{32}Pl + M_3\right) \tag{16}$$

By substituting equation (16) into equation (13) we get

$$\delta = \frac{7}{16} \times \frac{Pl^3}{48EI} + \frac{M_2 l^2}{16EI} = \frac{7}{16} \times \frac{Pl^3}{48EI} - \frac{l^2}{16EI} \times \left(\frac{9}{7 \times 32}Pl + M_3\right) = \frac{71}{224} \times \frac{Pl^3}{48EI} - \frac{M_3 l^2}{16EI} \tag{17}$$

Thus, by adding one set of beams on each side of the loaded beam, the deflection decreases to 0.4375 of its value with only the loaded beam, and by adding two sets of beams the deflection reduces to 0.3170 of its value with only the loaded beam.

4. <u>Third Unloaded Beams</u>

In this case there will be seven beams, three on each side of the loaded beam. For this case, the three-moment theorem gives

$$M_2 + 4M_3 + M_4 = 0 \tag{18}$$

Substituting for M_2 from equation (16) we get

$$-\frac{3}{7}\left(\frac{3}{32}Pl + M_3\right) + 4M_3 + M_4 = 0$$

or

$$M_3 = \frac{7}{25}\left(\frac{9}{224}Pl - M_4\right) \tag{19}$$

$$\begin{aligned}\delta &= \frac{71}{224} \times \frac{Pl^3}{48EI} - \frac{M_3 l^2}{16EI} = \frac{71}{224} \times \frac{Pl^3}{48EI} - \frac{l^2}{16EI} \times \frac{7}{25}\left(\frac{9}{224}Pl - M_4\right) \\ &= \frac{13}{50} \times \frac{Pl^3}{48EI} + \frac{7}{25} \times \frac{M_4 l^2}{16EI} = 0.260 \times \frac{Pl^3}{48EI} + \frac{7}{25} \times \frac{M_4 l^2}{16EI}\end{aligned} \tag{20}$$

5. Fourth Unloaded Beams

In this case there will be nine beams, four on each side of the loaded beam. For this case, the three-moment theorem gives

$$M_3 + 4M_4 + M_5 = 0 \tag{21}$$

Substituting for M_3 from equation (19) we get

$$\frac{7}{25}\left(\frac{9}{224}Pl - M_4\right) + 4M_4 + M_5 = 0$$

or

$$M_4 = -\frac{25}{93}\left(\frac{63}{5600}Pl + M_5\right) \tag{22}$$

Substituting into equation (20) gives

$$\delta = 0.260 \times \frac{Pl^3}{48EI} - \frac{7}{25} \times \frac{l^2}{16EI} \times \frac{25}{93}\left(\frac{63}{5600}Pl + M_5\right) = 0.2575 \times \frac{Pl^3}{48EI} - \frac{7}{16 \times 93}\frac{M_5 l^2}{EI}$$

6. Fifth Unloaded Beams

In this case there will be eleven beams, five on each side of the loaded beam. For this case, the three-moment theorem gives

$$M_4 + 4M_5 + M_6 = 0 \tag{23}$$

Substituting for M_4 from equation (22) we get

$$-\frac{25}{93}\left(\frac{63}{5600}Pl + M_5\right) + 4M_5 + M_6 = 0$$

or

$$M_5 = 0.2680\left(\frac{21}{31 \times 224}Pl - M_6\right) \tag{24}$$

$$\begin{aligned}\delta &= 0.2575 \times \frac{Pl^3}{48EI} - \frac{7l^2}{16 \times 93EI} \times 0.2680\left(\frac{21}{31 \times 224}Pl - M_6\right) \\ &= 0.2573 \times \frac{Pl^3}{48EI} + 0.00126\frac{M_6 l^2}{EI}\end{aligned} \tag{25}$$

7. Sixth Unloaded Beams

In this case there will be thirteen beams, six on each side of the loaded beam. For this case, the three-moment theorem gives

$$M_5 + 4M_6 + M_7 = 0 \tag{26}$$

Substituting for M_5 from equation (24) we get

$$0.2680\left(\frac{21}{31 \times 224} Pl - M_6\right) + 4M_6 + M_7 = 0$$

or

$$M_6 = -0.2680 \times (0.00081Pl + M_7) \tag{27}$$

$$\begin{aligned} \delta &= 0.2573 \times \frac{Pl^3}{48EI} - 0.00126\frac{l^2}{EI} \times 0.2680 \times (0.00081Pl + M_7) \\ &= 0.2573 \times \frac{Pl^3}{48EI} - 0.00034\frac{M_7 l^2}{EI} \end{aligned} \tag{28}$$

which agrees to four decimal places with the previous value.

# Beams	Reduction in δ
1	1
3	0.4375
5	0.3170
7	0.2600
9	0.2575
11	0.2573
13	0.2573

The Deflection at the End of a Continuous Beam

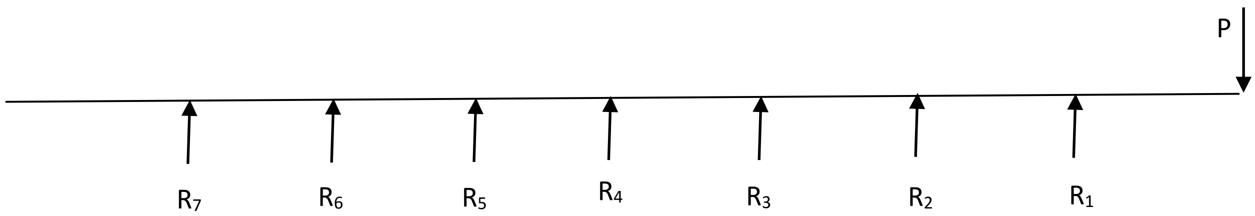


Figure C-1. A beam with many equally spaced supports loaded at the end.

Consider the continuous beam of Figure C-1 which is simply supported at equal spacing l and subject to a load P at the right end. We want to determine the deflection under the load. From statics we get two equations, one vertical force balance, and one moment-balance equation.

Thus

$$P = \sum_1^n R_i \quad \text{and} \quad P + \sum_1^n iR_{i+1} = 0 \tag{1}$$

<u>Solution in the first segment</u>

The differential equation for deflection in the first segment (to the right of R_1) and its solution, assuming $x = 0$ at R_1, is

$$\begin{aligned}
EIy'' &= P(l - x) \\
EIy' &= EIy_1' + P\left(lx - \frac{x^2}{2}\right) \\
EIy &= EIy_1'x + P\left(l\frac{x^2}{2} - \frac{x^3}{6}\right) \\
EIy(l) &= EIy_1'l + \frac{Pl^3}{3}
\end{aligned} \tag{2}$$

Thus, the desired deflection is

$$y(l) \equiv \Delta = y_1'l + \frac{Pl^3}{3EI} \tag{3}$$

In which the well-known solution for the cantilever beam is obtained by setting $y_1' = 0$.

Solution in the second segment

The differential equation for the deflection in the second segment of the beam and its solution, assuming that $x = 0$ at the beginning of the second segment, is

$$
\begin{aligned}
&EIy'' = P(2l - x) - R_1(l - x) \\
&EIy' = EIy_2' + P\left(2lx - \frac{x^2}{2}\right) - R_1\left(lx - \frac{x^2}{2}\right) \\
&EIy = EIy_2' x + P\left(2l\frac{x^2}{2} - \frac{x^3}{6}\right) - R_1\left(l\frac{x^2}{2} - \frac{x^3}{6}\right)
\end{aligned}
\tag{4}
$$

Evaluating equations (4) at $x = l$ we get

$$
\begin{aligned}
&EIy''(l) = Pl \\
&EIy'(l) = EIy_1' = EIy_2' + (3P - R_1)\frac{l^2}{2} \\
&EIy(l) = 0 = EIy_2' l + (5P - 2R_1)\frac{l^3}{6}
\end{aligned}
\tag{5}
$$

Substituting the third of equations (5) into the second, we get

$$
EIy_1' = -(5P - 2R_1)\frac{l^2}{6} + (3P - R_1)\frac{l^2}{2} = \left(2P - \frac{1}{2}R_1\right)\frac{l^2}{3} \tag{6}
$$

Substituting equation (6) into equation (3)

$$
\Delta = \frac{l^3}{3EI}\left(2P - \frac{1}{2}R_1 + P\right) = \frac{l^3}{3EI}\left(3P - \frac{1}{2}R_1\right) \tag{7}
$$

The Case of a Two-Segment Beam

In this case equations (1) reduce to

$$
\begin{aligned}
&R_1 + R_2 = P \\
&R_2 = -P \\
&\therefore R_1 = 2P
\end{aligned}
$$

Then from equation (7)

$$
\Delta = 2\frac{Pl^3}{3EI} = \frac{Pl^3}{1.5EI} \tag{8}
$$

which is twice the value for a cantilever beam.

Solution in the third segment

The differential equation for the deflection in the third segment of the beam and its solution, assuming that $x=0$ at the beginning of the third segment, is

$$
\begin{aligned}
EIy'' &= P\left(3l-x\right)-R_1\left(2l-x\right)-R_2\left(l-x\right)\\
EIy' &= EIy_3'+P\left(3lx-\frac{x^2}{2}\right)-R_1\left(2lx-\frac{x^2}{2}\right)-R_2\left(lx-\frac{x^2}{2}\right)\\
EIy &= EIy_3'x+P\left(3l\frac{x^2}{2}-\frac{x^3}{6}\right)-R_1\left(2l\frac{x^2}{2}-\frac{x^3}{6}\right)-R_2\left(l\frac{x^2}{2}-\frac{x^3}{6}\right)
\end{aligned}
\tag{9}
$$

Evaluating equations (9) at $x=l$ we get

$$
\begin{aligned}
EIy''(l) &= \left(2P-R_1\right)l\\
EIy'(l) &= EIy_2' = EIy_3'+\left(5P-3R_1-R_2\right)\frac{l^2}{2}\\
EIy\left(l\right) &= 0 = EIy_3'l+\left(8P-5R_1-2R_2\right)\frac{l^3}{6}
\end{aligned}
\tag{10}
$$

Substituting the third of equations (10) into the second we get

$$
\begin{aligned}
EIy_2' &= -\left(8Pl^2-5R_1l^2-2R_2\right)\frac{l^2}{6}+\left(5P-3R_1-R_2\right)\frac{l^2}{2}\\
&= \left(7P-4R_1-R_2\right)\frac{l^2}{6}
\end{aligned}
\tag{11}
$$

Substituting equation (11) into the third of equations (5)

$$
\begin{aligned}
&0 = EIy_2'+\left(5P-2R_1\right)\frac{l^2}{6} = \left(7P-4R_1-R_2+5P-2R_1\right)\frac{l^2}{6}\\
&\text{or}\\
&6R_1+R_2 = 12P
\end{aligned}
\tag{12}
$$

The Case of a Three-Segment Beam

If there are only three segments before the beam ends, equations (1) become

$$R_1 + R_2 + R_3 = P$$
$$R_2 + 2R_3 = -P$$

Multiply the first of these equations by 2 and subtract from the second. The result is

$$2R_1 + R_2 = 3P$$

Subtracting this equation from equation (12) gives

$$R_1 = \frac{9}{4}P$$

Substituting this value of R_1 into equation (7) gives

$$\Delta = \frac{Pl^3}{3EI}\left(3 - \frac{9}{8}\right) = \frac{15}{8}\frac{Pl^3}{3EI} = \frac{Pl^3}{1.6EI} \tag{13}$$

Solution in the fourth segment

If there is a fourth segment, the differential equation for the deflection of the fourth segment and its solution, assuming that $x = 0$ at the beginning of the fourth segment, is

$$\begin{aligned} EIy'' &= P(4l - x) - R_1(3l - x) - R_2(2l - x) - R_3(l - x) \\ EIy' &= EIy_4' + P\left(4lx - \frac{x^2}{2}\right) - R_1\left(3lx - \frac{x^2}{2}\right) - R_2\left(2lx - \frac{x^2}{2}\right) - R_3\left(lx - \frac{x^2}{2}\right) \\ EIy &= EIy_4'x + P\left(4l\frac{x^2}{2} - \frac{x^3}{6}\right) - R_1\left(3l\frac{x^2}{2} - \frac{x^3}{6}\right) - R_2\left(2l\frac{x^2}{2} - \frac{x^3}{6}\right) - R_3\left(l\frac{x^2}{2} - \frac{x^3}{6}\right) \end{aligned} \tag{14}$$

Evaluating equations (14) at $x = l$ we get

$$\begin{aligned} EIy''(l) &= (3P - 2R_1 - R_2)l \\ EIy'(l) &= EIy_3' = EIy_4' + (7P - 5R_1 - 3R_2 - R_3)\frac{l^2}{2} \\ EIy(l) &= 0 = EIy_4'l + (11P - 8R_1 - 5R_2 - 2R_3)\frac{l^3}{6} \end{aligned} \tag{15}$$

Substituting the third of equations (15) into the second gives

$$\begin{aligned} EIy_3' &= -(+11P - 8R_1 - 5R_2 - 2R_3)\frac{l^2}{6} + (7P - 5R_1 - 3R_2 - R_3)\frac{l^2}{2} \\ &= (10P - 7R_1 - 4R_2 - R_3)\frac{l^2}{6} \end{aligned} \tag{16}$$

Substituting equation (16) into the third of equations (10) gives

$$EIy_3^{'} + \left(8P - 5R_1 - 2R_2\right)\frac{l^2}{6} = \left(10P - 7R_1 - 4R_2 - R_3 + 8P - 5R_1 - 2R_2\right)\frac{l^2}{6} = 0$$

or (17)

$$18P - 12R_1 - 6R_2 - R_3 = 0$$

<u>The Case of a Four-Segment Beam</u>

If the beam ends at R_4 we have, from equations (1), (12) and (17), the following four equations for the four reactions:

$$\begin{aligned} R_1 + R_2 + R_3 + R_4 &= P \\ R_2 + 2R_3 + 3R_4 &= -P \\ 6R_1 + R_2 &= 12P \\ 12R_1 + 6R_2 + R_3 &= 18P \end{aligned} \tag{18}$$

Multiply the first of these equations by 3 and subtract it from the second. The result is

$$3R_1 + 2R_2 + R_3 = 4P$$

Subtract this equation from the fourth of equations (18). The result is

$$9R_1 + 4R_2 = 14P \tag{19}$$

Multiply the third of equations (18) by four and subtract equation (19). The result is

$$R_1 = \frac{34}{15}P$$

Substitution of this value of R_1 into equation (7) gives

$$\Delta = \frac{l^3}{3EI}\left(3P - \frac{1}{2}\times\frac{34}{15}P\right) = \frac{28}{15}\frac{Pl^3}{3EI} = \frac{Pl^3}{1.607EI} \tag{20}$$

<u>Solution in the fifth segment</u>

The differential equation for the deflection of the fifth segment of the beam and its solution, assuming that $x = 0$ at the beginning of the fifth segment, is

$$EIy'' = P\left(5l - x\right) - R_1\left(4l - x\right) - R_2\left(3l - x\right) - R_3\left(2l - x\right) - R_4\left(l - x\right)$$
$$EIy' = EIy_5' + P\left(5lx - \frac{x^2}{2}\right) - R_1\left(4lx - \frac{x^2}{2}\right) - R_2\left(3lx - \frac{x^2}{2}\right) - R_3\left(2lx - \frac{x^2}{2}\right) - R_4\left(lx - \frac{x^2}{2}\right)$$
$$EIy = EIy_5'x + P\left(5l\frac{x^2}{2} - \frac{x^3}{6}\right) - R_1\left(4l\frac{x^2}{2} - \frac{x^3}{6}\right) - R_2\left(3l\frac{x^2}{2} - \frac{x^3}{6}\right) - R_3\left(2l\frac{x^2}{2} - \frac{x^3}{6}\right) - R_4\left(l\frac{x^2}{2} - \frac{x^3}{6}\right)$$
(21)

Evaluating equations (21) at $x = l$ we get

$$EIy''(l) = \left(4P - 3R_1 - 2R_2 - R_3\right)l$$
$$EIy'(l) = EIy_4' = EIy_5' + \left(9P - 7R_1 - 5R_2 - 3R_3 - R_4\right)\frac{l^2}{2}$$
$$EIy\left(l\right) = 0 = EIy_5'l + \left(14P - 11R_1 - 8R_2 - 5R_3 - 2R_4\right)\frac{l^3}{6}$$
(22)

Substituting the third of equations (22) into the second gives

$$\begin{aligned} EIy_4' &= -\left(+14P - 11R_1 - 8R_2 - 5R_3 - 2R_4\right)\frac{l^2}{6} + \left(9P - 7R_1 - 5R_2 - 3R_3 - R_4\right)\frac{l^2}{2} \\ &= \left(13P - 10R_1 - 7R_2 - 4R_3 - R_4\right)\frac{l^2}{6} \end{aligned} \tag{23}$$

Substituting equation (23) into the third of equations (15) gives

$$EIy_4' + \left(11P - 8R_1 - 5R_2 - 2R_3\right)\frac{l^2}{6} = \left(13P - 10R_1 - 7R_2 - 4R_3 - R_4 + 11P - 8R_1 - 5R_2 - 2R_3\right)\frac{l^2}{6} = 0$$

or

$$24P - 18R_1 - 12R_2 - 6R_3 - R_4 = 0$$
(24)

The Case of a Five-Segment Beam

If the beam ends at R_5 we have, from equations (1), (12), (17) and (24), the following five equations for the five reactions:

$$
\begin{aligned}
&R_1 + R_2 + R_3 + R_4 + R_5 = P \\
&R_2 + 2R_3 + 3R_4 + 4R_5 = -P \\
&6R_1 + R_2 = 12P \\
&12R_1 + 6R_2 + R_3 = 18P \\
&18R_1 + 12R_2 + 6R_3 + R_4 = 24P
\end{aligned}
\tag{25}
$$

Multiply the first of these equations by 4 and subtract it from the second. The result is

$$4R_1 + 3R_2 + 2R_3 + R_4 = 5P$$

Subtract this equation from the fifth of equations (25). The result is

$$14R_1 + 9R_2 + 4R_3 = 19P \tag{26}$$

Multiply the fourth of equations (25) by four and subtract equation (26). The result is

$$34R_1 + 15R_2 = 53P \tag{27}$$

Multiply the third of equations (25) by 15 and subtract equation (27). The result is

$$R_1 = \frac{127}{56}P$$

Then equation (7) becomes

$$\Delta = \frac{l^3}{3EI}\left(3P - \frac{1}{2}\times\frac{127}{56}P\right) = \frac{209}{112}\frac{Pl^3}{3EI} = \frac{Pl^3}{1.60766EI} \tag{28}$$

<u>The Case of a Six-Segment Beam</u>

From equation (25) we can see the pattern, which permits us to write down the equations for the reactions in the six-segment beam without the tedium of solving the differential equations. Thus, the reaction equations for the six-segment beam are

$$
\begin{aligned}
&R_1 + R_2 + R_3 + R_4 + R_5 + R_6 = P \\
&R_2 + 2R_3 + 3R_4 + 4R_5 + 5R_6 = -P \\
&6R_1 + R_2 = 12P \\
&12R_1 + 6R_2 + R_3 = 18P \\
&18R_1 + 12R_2 + 6R_3 + R_4 = 24P \\
&24R_1 + 18R_2 + 12R_3 + 6R_4 + R_5 = 30P
\end{aligned}
\tag{29}
$$

Thus, multiply the first of equations (29) by 5 and subtract the second. The result is

$$5R_1 + 4R_2 + 3R_3 + 2R_4 + R_5 = 6P \tag{30}$$

Subtract equation (30) from the last of equations (29). The result is

$$19R_1 + 14R_2 + 9R_3 + 4R_4 = 24P \tag{31}$$

Multiply the second from the last of equations (29) by 4 and subtract equation (31). The result is

$$53R_1 + 34R_2 + 15R_3 = 72P \tag{32}$$

Multiply the 4th of equations (29) by 15 and subtract equation (32). The result is

$$127R_1 + 56R_2 = 198P \tag{33}$$

Substitute for R_2 in equation (33) from the third of equations (29). The result is

$$\begin{aligned} &127R_1 + 56\left(12P - 6R_1\right) = 198P \\ &\therefore R_1 = \frac{198 - 56\cdot 12}{56\cdot 6 - 127}P = \frac{474}{209}P \end{aligned}$$

Substitute this value of R_1 into equation (7) to give

$$\Delta = \frac{l^3}{3EI}\left(3P - \frac{1}{2}\times\frac{474}{209}P\right) = \frac{780}{418}\frac{Pl^3}{3EI} = \frac{Pl^3}{1.60769EI} \tag{34}$$

So, we see that we can take for the end deflection of a beam with an indefinite number of segments as

$$\boxed{\Delta = 1.866\frac{Pl^3}{3EI} = \frac{Pl^3}{1.608EI}} \tag{35}$$

which is the information sought by this analysis, and is close to the Fibonacci ratio 1.618.

The Deflection of a Curved Beam

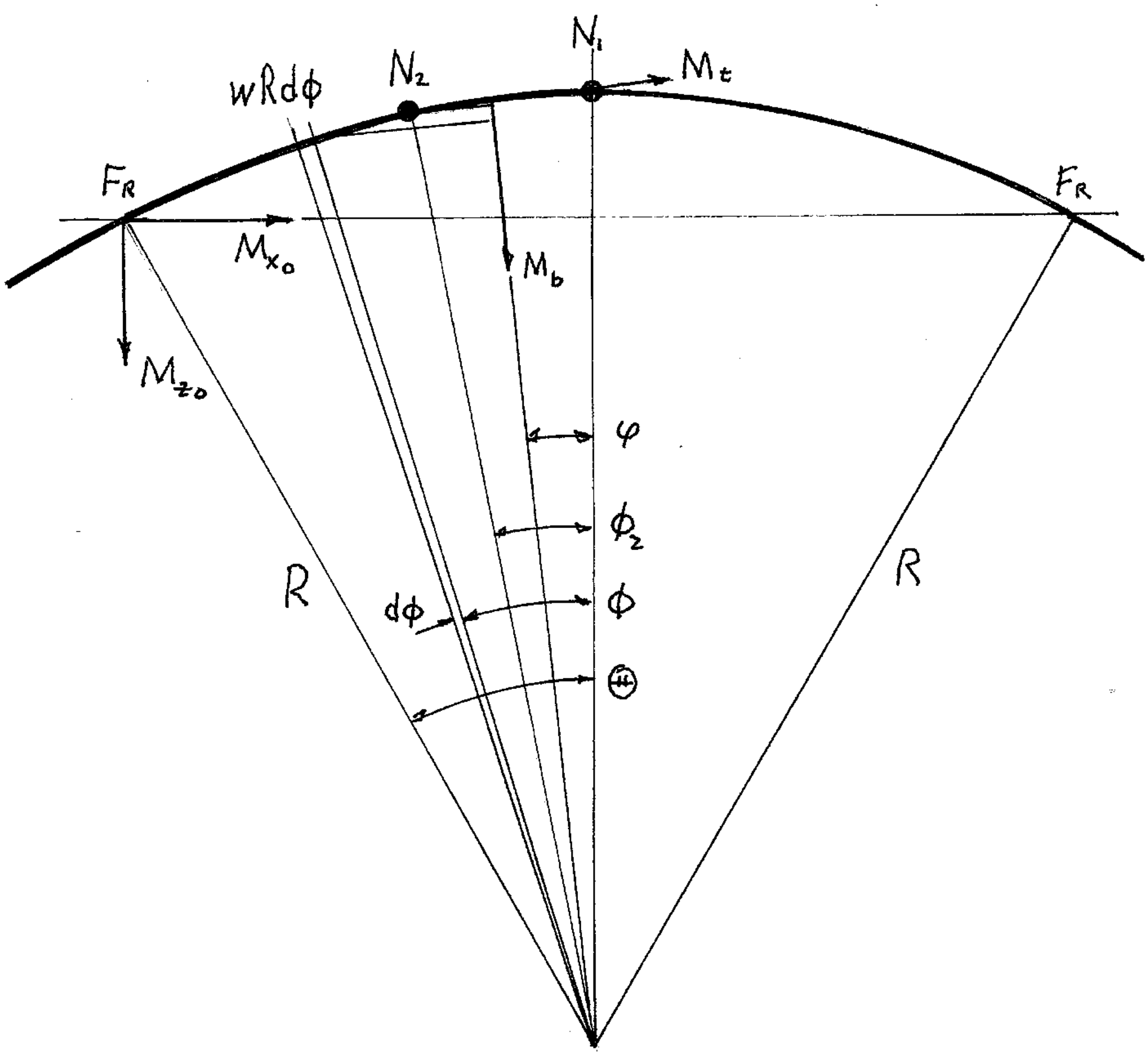


Figure 1. The Notation used in Analysis of a Curved Beam.

1. The Problem and the Solution Method

Consider the beam diagrammed in Figure 1 of length *L* between supports and curved to a radius *R*. The beam is clamped at both ends and is subject to a uniform load *w* per unit of length applied downward and three point loads representing vehicles. One of these vehicles, of weight N_1 is at the center of the span, and a pair of vehicles of weight N_2 is spaced equally distance to the left and right of the center of the span. Only the vehicle on the left is shown. The distance between these vehicles is VT_h, where *V* is the vehicle's speed and T_h is the time headway. Additional vehicles at distances from mid-span of 2, 3, 4, etc. times VT_h can be added later. This arrangement preserves the symmetry of the solution about the midpoint.

The objective of this analysis is to determine the deflection at the center of the guideway under the vehicle load N_1. Figure 1 shows the notation used. An orthogonal x-y-z reference frame has its origin at the left post, with the x-axis extending to the right through the right post, y downward in the direction of the deflection we wish to calculate, and z with a right-handed reference frame points horizontally and upward in the figure. The radius line from the center of curvature to the left post and the radius line at mid-span are separated by the angle

$$\Theta = \frac{L}{2R}. \tag{1.1}$$

The vehicle weight N_2 is at the angular position ϕ_{N_2} from the centerline, where $\phi_{N_2} = iVT_h / R = iD_v / R,$ where *i* is the number of headway length from the center point. A differential element of the guideway of weight $wRd\phi$ at the angular distance ϕ from the centerline is shown and an arbitrary point φ along the guideway is shown. From statics, the upward reaction force at each post is

$$F_R = \frac{1}{2}\left(wL + N_1\right) + N_2 = wR\Theta + \frac{1}{2}N_1 + N_2 \tag{1.2}$$

We will neglect the centrifugal force produced by the motion of each vehicle because 1) it is limited to the unbalanced fraction of the weight of each car, 2) the lateral acceleration is limited by ride comfort to the value

$$a_H = g\tan\phi + a_l / \cos\phi = g\left(\tan 6^o + 0.2 / \cos 6^o\right) = 0.31g \tag{1.3}$$

and 3) because the lever arm for its moment is much smaller than the lever arm due to the vehicle weight acting downward

The solution method is derived by Timoshenko.[16] It is to apply Castigliano's 1875 Theorem, which says that the partial derivative of the strain energy *U* in a beam with respect to a point load *P* is the deflection δ under the load, i.e.,

$$\frac{\partial U}{\partial P} = \delta \tag{1.4}$$

Conveniently, we place a vehicle of weight N_1 at the center point where we want the deflection.

Note from Figure 1 that at the clamped left end of the beam the moment vector is indicated by a horizontal component M_{xo} in the direction of the x-axis and a horizontal and transverse component M_{zo}. Both must be positive in the directions shown to balance the loads on the guideway. M_{xo} is determined by statics, but M_{zo} is statically indeterminate, and must be determined by the principle of least work[17], which for a clamped, non-moving support requires that

[16] S. Timoshenko, *Strength of Materials*, Part I, Elementary Theory and Problems, Section 69, The Theorem of Castigliano , pp. 308-320.

[17] Ibid. Section 70. Application of Castigliano's Theorem in Solution of Statically Indeterminate Problems, pp. 320-330.

$$\frac{\partial U}{\partial M_{zo}} = 0 \tag{1.5}$$

As a guide, the corresponding solution for a straight beam is given in Section 6.

$$\int_0^{\Theta} \sin\varphi d\varphi = 1-\cos\Theta, \quad \int_0^{\Theta} \cos\varphi d\varphi = \sin\Theta, \quad \int_0^{\Theta} \sin\varphi\cos\varphi d\varphi = \frac{1}{2}\sin^2\Theta$$

$$\int_0^{\Theta} \varphi\sin\varphi d\varphi = \sin\Theta - \Theta\cos\Theta, \quad \int_0^{\Theta} \varphi\cos\varphi d\varphi = \cos\Theta + \Theta\sin\Theta - 1$$

$$\int_0^{\Theta} \cos^2\varphi d\varphi = \frac{1}{2}\left(\Theta + \sin\Theta\cos\Theta\right), \quad \int_0^{\Theta} \sin^2\varphi d\varphi = \frac{1}{2}\left(\Theta - \sin\Theta\cos\Theta\right)$$

$$\int_0^{\Theta} \sin\left(\Theta-\varphi\right) d\varphi = 1-\cos\Theta, \quad \int_0^{\Theta} \cos\left(\Theta-\varphi\right) d\varphi = \sin\Theta$$

$$\int_0^{\Theta} \sin\left(\Theta-\varphi\right)\cos\varphi d\varphi = \int_0^{\Theta} \cos\left(\Theta-\varphi\right)\sin\varphi d\varphi = \frac{1}{2}\Theta\sin\Theta$$

$$\int_0^{\Theta} \cos\left(\Theta-\varphi\right)\cos\varphi d\varphi = \frac{1}{2}\left(\sin\Theta + \Theta\cos\Theta\right)$$

$$\int_0^{\Theta} \sin\left(\Theta-\varphi\right)\sin\varphi d\varphi = \frac{1}{2}\left(\sin\Theta - \Theta\cos\Theta\right)$$

Figure 2. The Integrals Used.

2. The Moments

By examining Figure 1 we see that the statically determinant moment is given by the equation

$$\begin{aligned} M_{xo} &= \int_0^{\Theta} wRd\phi \cdot R\left[\cos\phi - \cos\Theta\right] + \frac{1}{2}N_1R(1-\cos\Theta) + N_2R\left(\cos\phi_{N_2} - \cos\Theta\right) \\ &= wR^2(\sin\Theta - \Theta\cos\Theta) + \frac{1}{2}N_1R(1-\cos\Theta) + N_2R\left(\cos\phi_{N_2} - \cos\Theta\right) \end{aligned} \tag{2.1}$$

When $R \to \infty$ the beam straightens and from equation (1.1) $\Theta \to 0$. The expansion of equation (2.1) into the first two terms of its infinite series is

$$\begin{aligned} M_{xo} &\to wR^2\left(\Theta - \frac{\Theta^3}{6} - \Theta + \frac{\Theta^3}{2}\right) + \frac{1}{2}N_1R\left(1-1+\frac{\Theta^2}{2}\right) + N_2R\left(1-\frac{\phi_{N_2}^2}{2} - 1 + \frac{\Theta^2}{2}\right) \\ &= wR^2\frac{\Theta^3}{3} + N_1R\frac{\Theta^2}{4} + N_2R\frac{\left(\Theta^2 - \phi_{N_2}^2\right)}{2} \\ &= \frac{L}{2R}\left[\frac{wL^2}{12} + \frac{N_1L}{8} + \frac{N_2L}{4}\left(1-\frac{4D_v^2}{L^2}\right)\right] \to 0 \text{ as } R \to \infty. \end{aligned} \tag{2.2}$$

in which $D_v = VT_h$ is the above-mentioned distance between vehicles, and equation (1.1) has been used. We expected this result because as $\Theta \to 0$ M_{xo} becomes a purely torsional moment and vanishes since in a straight beam the applied forces have no torsional component. From Section 6 the term in brackets in the right-hand expression of equation (2.2) is the bending moment at each end of a straight clamped-clamped beam subject to a uniform load *w* and the indicated point loads. For M_{zo} the limit result must reduce to this quantity as $\Theta \to 0$, which we will see below.

From Figure 1 the bending moment M_b at an angular distance φ from the center point of the guideway is

$$\begin{aligned} M_b(\varphi) &= M_{xo}\sin\varphi + M_{zo}\cos\varphi - F_R R\sin(\Theta-\varphi) \\ &\quad + \int_{\varphi}^{\Theta} wRd\phi \cdot R\sin(\phi-\varphi) + N_2 R\sin\left\langle \phi_{N_2} - \varphi \right\rangle \\ &= M_{xo}\sin\varphi + M_{zo}\cos\varphi - F_R R\sin(\Theta-\varphi) \\ &\quad + wR^2\left[1-\cos(\Theta-\varphi)\right] + N_2 R\sin\left\langle \phi_{N_2} - \varphi \right\rangle \end{aligned} \tag{2.3}$$

in which if $Q \le 0,\ \langle Q \rangle = 0.$ Note that for small Θ and taking into account that the coordinate along the beam is then $x = R(\Theta-\varphi)$, equation (2.3) reduces to equation (6.1).

The torsional moment M_t at any point φ is given by

$$\begin{aligned} M_t(\varphi) &= M_{xo}\cos\varphi - M_{zo}\sin\varphi - F_R R\left[1-\cos(\Theta-\varphi)\right] + \\ &\quad \int_{\varphi}^{\Theta} wRd\phi \cdot R[1-\cos(\phi-\varphi)] + N_2 R\left[1-\cos\left\langle \phi_{N_2} - \varphi \right\rangle\right] \\ &= M_{xo}\cos\varphi - M_{zo}\sin\varphi - F_R R\left[1-\cos(\Theta-\varphi)\right] + \\ &\quad wR^2\left[\Theta-\varphi-\sin(\Theta-\varphi)\right] + N_2 R\left[1-\cos\left\langle \phi_{N_2} - \varphi \right\rangle\right] \end{aligned} \tag{2.4}$$

3. Calculation of the Statically Indeterminate Moment

The strain energy in the beam is given by the equation[18]

$$U = 2\int_0^{\Theta} \left(\frac{M_b^2}{2EI} + \frac{M_t^2}{2GI_p} \right) Rd\varphi \tag{3.1}$$

in which EI and GI_p are the bending and torsional rigidities, respectively. From equation (1.5), the principal of least work leads to

[18] Ibid. Chapter X, "Energy of Strain."

$$\int_0^{\Theta}\left(M_b\frac{\partial M_b}{\partial M_{zo}}+\beta M_t\frac{\partial M_t}{\partial M_{zo}}\right)d\varphi=\int_0^{\Theta}\left(M_b\cos\varphi-\beta M_t\sin\varphi\right)d\varphi=0 \tag{3.2}$$

in which the derivatives come from equations (2.3) and (2.4) and in which

$$\beta\equiv\frac{EI}{GI_p} \tag{3.3}$$

Substitute equations (2.3) and (2.4) into equation (3.2). We get

$$\begin{aligned}
&M_{xo}\int_0^{\Theta}\sin\varphi\cos\varphi d\varphi+M_{zo}\int_0^{\Theta}\cos^2\varphi d\varphi-F_RR\int_0^{\Theta}\sin\left(\Theta-\varphi\right)\cos\varphi d\varphi\\
&+wR^2\int_0^{\Theta}\left[1-\cos\left(\Theta-\varphi\right)\right]\cos\varphi d\varphi+N_2R\int_0^{\phi_{N_2}}\sin\left(\phi_{N_2}-\varphi\right)\cos\varphi d\varphi\\
&=\beta M_{xo}\int_0^{\Theta}\sin\varphi\cos\varphi d\varphi-\beta M_{zo}\int_0^{\Theta}\sin^2\varphi d\varphi-\beta F_RR\int_0^{\Theta}\left[1-\cos\left(\Theta-\varphi\right)\right]\sin\varphi d\varphi\\
&+\beta wR^2\int_0^{\Theta}\left[\Theta-\varphi-\sin\left(\Theta-\varphi\right)\right]\sin\varphi d\varphi+\beta N_2R\int_0^{\phi_{N_2}}\left[1-\cos\left(\phi_{N_2}-\varphi\right)\right]\sin\varphi d\varphi
\end{aligned} \tag{3.4}$$

Perform the integrations per Figure B-2, multiply by 2, and introduce without loss of generality the notation

$$S\Theta\equiv\sin\Theta,\quad C\Theta=\cos\Theta \tag{3.5}$$

Then, with cancelations where needed,

$$\begin{aligned}
&M_{xo}S^2\Theta+M_{zo}\left(\Theta+S\Theta C\Theta\right)-F_RR\Theta S\Theta+wR^2\left(S\Theta-\Theta C\Theta\right)+N_2R\phi_{N_2}S\phi_{N_2}\\
&=\beta M_{xo}S^2\Theta-\beta M_{zo}\left(\Theta-S\Theta C\Theta\right)-\beta F_RR\left[2\left(1-C\Theta\right)-\Theta S\Theta\right]\\
&+\beta wR^2\left(2\Theta+\Theta C\Theta-3S\Theta\right)+\beta N_2R\left[2\left(1-C\phi_{N_2}\right)-\phi_{N_2}S\phi_{N_2}\right]
\end{aligned} \tag{3.6}$$

Solve for M_{zo}, the statically indeterminate moment:

$$M_{zo}=\frac{\begin{pmatrix}\left(\beta-1\right)M_{xo}S^2\Theta-\beta F_RR\left\{\beta\left[2\left(1-C\Theta\right)-\Theta S\Theta\right]-\Theta S\Theta\right\}\\ +wR^2\left[\beta\left(2\Theta+\Theta C\Theta-3S\Theta\right)-S\Theta+\Theta C\Theta\right]\\ +N_2R\left\{\beta\left[2\left(1-C\phi_{N_2}\right)-\phi_{N_2}S\phi_{N_2}\right]-\phi_{N_2}S\phi_{N_2}\right\}\end{pmatrix}}{\Theta+S\Theta C\Theta+\beta\left(\Theta-S\Theta C\Theta\right)} \tag{3.7}$$

Substitute for F_RR from equation. Then

$$M_{zo}=\frac{\begin{pmatrix}(\beta-1)M_{xo}S^2\Theta\\ +wR^2\left[\beta(2\Theta+\Theta C\Theta-3S\Theta)-S\Theta+\Theta C\Theta+\Theta\left\{\beta\left[\Theta S\Theta-2(1-C\Theta)\right]+\Theta S\Theta\right\}\right]\\ +\frac{1}{2}N_1R\left\{\beta\left[\Theta S\Theta-2(1-C\Theta)\right]+\Theta S\Theta\right\}\\ +N_2R\left\{\beta\left[\Theta S\Theta-2(1-C\Theta)\right]-\beta\left[\phi_{N_2}S\phi_{N_2}-2\left(1-C\phi_{N_2}\right)\right]+\Theta S\Theta-\phi_{N_2}S\phi_{N_2}\right\}\end{pmatrix}}{\Theta+S\Theta C\Theta+\beta(\Theta-S\Theta C\Theta)} \tag{3.8}$$

Substitute for M_{xo} from equation. Then

$$M_{zo}=\frac{\begin{pmatrix}+wR^2\begin{bmatrix}\beta(2\Theta+\Theta C\Theta-3S\Theta)-S\Theta+\Theta C\Theta\\ +\Theta\left\{\beta\left[\Theta S\Theta-2(1-C\Theta)\right]+\Theta S\Theta\right\}+(\beta-1)S^2\Theta(S\Theta-\Theta C\Theta)\end{bmatrix}\\ +\frac{1}{2}N_1R\left\{\beta\left[\Theta S\Theta-2(1-C\Theta)\right]+\Theta S\Theta+(\beta-1)S^2\Theta(1-C\Theta)\right\}\\ +N_2R\begin{Bmatrix}\beta\left[\Theta S\Theta-2(1-C\Theta)-\phi_{N_2}S\phi_{N_2}+2\left(1-C\phi_{N_2}\right)\right]\\ +\Theta S\Theta-\phi_{N_2}S\phi_{N_2}+(\beta-1)S^2\Theta\left(C\phi_{N_2}-C\Theta\right)\end{Bmatrix}\end{pmatrix}}{\Theta+S\Theta C\Theta+\beta(\Theta-S\Theta C\Theta)} \tag{3.9}$$

$$M_{zo}=C_{Mw}wR^2+\frac{1}{2}C_{MN_1}N_1R+C_{MN_2}N_2R \tag{3.10}$$

in which

$$C_{Mw}=\frac{(\beta+1)\Theta^2S\Theta-\left[2(\beta+1)+(\beta-1)C^2\Theta\right](S\Theta-\Theta C\Theta)}{\Theta+S\Theta C\Theta+\beta(\Theta-S\Theta C\Theta)}$$

$$C_{MN_1}=\frac{\beta\left[\Theta S\Theta-\left(1+C^2\Theta\right)(1-C\Theta)\right]+S\Theta\left[\Theta-S\Theta(1-C\Theta)\right]}{\left[\Theta+S\Theta C\Theta+\beta(\Theta-S\Theta C\Theta)\right]}$$

$$C_{MN_2}=\frac{(\beta+1)\left(\Theta S\Theta-\phi_{N_2}S\phi_{N_2}\right)-\left[\beta\left(1+C^2\Theta\right)+S^2\Theta\right]\left(C\phi_{N_2}-C\Theta\right)}{\Theta+S\Theta C\Theta+\beta(\Theta-S\Theta C\Theta)}$$

Following equation (2.2), consider the series expansion of equation (3.4) as Θ becomes very small.

$$C_{Mw}\to\frac{\Theta^3\{3\beta+3-2\beta-2-\beta+1\}}{6\Theta}=\frac{\Theta^2}{6}(2)=\frac{1}{3}\Theta^2$$

$$C_{MN_1}\to\frac{\Theta^2\left[\beta(1-1)+1\right]}{2\Theta}=\frac{1}{2}\Theta$$

$$C_{MN_2} \to \frac{\Theta^2\left[\beta\left(1-\left(\frac{\phi_{N_2}}{\Theta}\right)^2\right)+\left(1-\left(\frac{\phi_{N_2}}{\Theta}\right)^2\right)-\beta\left(1-\left(\frac{\phi_{N_2}}{\Theta}\right)^2\right)\right]}{2\Theta} = \frac{1}{2}\Theta\left[1-\left(\frac{\phi_{N_2}}{\Theta}\right)^2\right]$$

Since

$$R\Theta = \frac{L}{2} \quad \text{and} \quad R\phi_{N_2} = D_v$$

M_{zo} reduces to

$$\begin{aligned} M_{zo} &\to \frac{1}{3}w(R\Theta)^2 + \frac{1}{4}N_1R\Theta + \frac{1}{2}N_2R\Theta\left[1-\left(\frac{2D_v}{L}\right)^2\right] \\ &= \frac{wL^2}{12} + \frac{N_1L}{8} + \frac{N_2L}{4}\left(1-\frac{4D_v^2}{L^2}\right) \end{aligned} \tag{3.11}$$

which agrees with the solution for a straight beam given by equation (6.4) and for the N_2 term with equation (7.9). Compare also with equation (2.2).

4. Calculation of the Deflection

The deflection at the center of the beam can be found from equation (1.4), in which we set $P = N_1$. Note that N_1 appears only in the terms M_{xo}, M_{zo}, and F_R. Therefore, we can write

$$\begin{aligned} U &= U\left[M_{xo}(N_1), M_{zo}(N_1), F_R(N_1)\right] \\ \delta &= \frac{\partial U}{\partial N_1} = \frac{\partial U}{\partial M_{xo}}\frac{\partial M_{xo}}{\partial N_1} + \frac{\partial U}{\partial M_{zo}}\frac{\partial M_{zo}}{\partial N_1} + \frac{\partial U}{\partial F_R}\frac{\partial F_R}{\partial N_1} \end{aligned} \tag{4.1}$$

Taking into account equation (1.5), this equation reduces to

$$\delta = \frac{\partial U}{\partial N_1} = \frac{\partial U}{\partial M_{xo}}\frac{\partial M_{xo}}{\partial N_1} + \frac{\partial U}{\partial F_R}\frac{\partial F_R}{\partial N_1} \tag{4.2}$$

The strain energy U is given by equation (3.1). Using equation (3.3) the differential of U can be written in the form

$$\therefore \delta = \frac{\partial U}{\partial N_1} = \frac{2R}{EI}\int_0^\Theta \left[M_b\left(\frac{\partial M_b}{\partial M_{xo}}\frac{\partial M_{xo}}{\partial N_1} + \frac{\partial M_b}{\partial F_R}\frac{\partial F_R}{\partial N_1}\right) + \beta M_t\left(\frac{\partial M_t}{\partial M_{xo}}\frac{\partial M_{xo}}{\partial N_1} + \frac{\partial M_t}{\partial F_R}\frac{\partial F_R}{\partial N_1}\right)\right]d\varphi \tag{4.3}$$

in which, from equations (2.3) and (2.4) respectively we have

$$
\begin{aligned}
&\frac{\partial M_b}{\partial M_{xo}} = S\varphi, \quad \frac{\partial M_b}{\partial F_R} = -RS\left(\Theta-\varphi\right), \\
&\frac{\partial M_t}{\partial M_{xo}} = C\varphi, \quad \frac{\partial M_t}{\partial F_R} = -R\left[1-C\left(\Theta-\varphi\right)\right]
\end{aligned}
\tag{4.4}
$$

and from equations (2.1) and (1.2) respectively, we have

$$
\frac{\partial M_{xo}}{\partial N_1} = \frac{R}{2}\left(1-C\Theta\right), \quad \frac{\partial F_R}{\partial N_1} = \frac{1}{2}
\tag{4.5}
$$

Thus

$$
\begin{aligned}
\delta &= \frac{R^2}{EI}\int_0^{\Theta}\left\{M_b\left[S\varphi\left(1-C\Theta\right)-S\left(\Theta-\varphi\right)\right]+\beta M_t\left[C\varphi\left(1-C\Theta\right)-1+C\left(\Theta-\varphi\right)\right]\right\}d\varphi \\
&= \frac{R^2}{EI}\int_0^{\Theta}\left[M_b\left(S\varphi-S\Theta C\varphi\right)+\beta M_t\left(C\varphi-1+S\Theta S\varphi\right)\right]d\varphi
\end{aligned}
\tag{4.6}
$$

Substituting equations (2.3) and (2.4) into equation (4.6) we get

$$
\begin{aligned}
\delta &= \frac{M_{xo}R^2}{EI}\int_0^{\Theta}\left\{S\varphi\left(S\varphi-S\Theta C\varphi\right)+\beta\left[C\varphi\left(C\varphi-1+S\Theta S\varphi\right)\right]\right\}d\varphi \\
&+\frac{M_{zo}R^2}{EI}\int_0^{\Theta}\left\{C\varphi\left(S\varphi-S\Theta C\varphi\right)-\beta\left[S\varphi\left(C\varphi-1+S\Theta S\varphi\right)\right]\right\}d\varphi \\
&-\frac{F_R R^3}{EI}\int_0^{\Theta}\left\{S\left(\Theta-\varphi\right)\left(S\varphi-S\Theta C\varphi\right)+\beta\left[\left[1-C\left(\Theta-\varphi\right)\right]\left(C\varphi-1+S\Theta S\varphi\right)\right]\right\}d\varphi \\
&+\frac{wR^4}{EI}\int_0^{\Theta}\left\{\left[1-C\left(\Theta-\varphi\right)\right]\left(S\varphi-S\Theta C\varphi\right)+\beta\left[\left[\Theta-\varphi-S\left(\Theta-\varphi\right)\right]\left(C\varphi-1+S\Theta S\varphi\right)\right]\right\}d\varphi \\
&+\frac{N_2R^3}{EI}\int_0^{\phi_{N_2}}\left\{S\left(\phi_{N_2}-\varphi\right)\left(S\varphi-S\Theta C\varphi\right)+\beta\left[\left[1-C\left(\phi_{N_2}-\varphi\right)\right]\left(C\varphi-1+S\Theta S\varphi\right)\right]\right\}d\varphi
\end{aligned}
$$

Integrating, using the integrals of Figure B-2, after some reduction we get

$$
\delta = \frac{M_{xo}R^2}{2EI}C_x+\frac{M_{zo}R^2}{2EI}C_z+\frac{F_R R^3}{2EI}C_F+\frac{wR^4}{2EI}C_w+\frac{N_2R^3}{2EI}C_{N_2}
\tag{4.7}
$$

in which

$$C_x = \Theta - S\Theta C\Theta - S^3\Theta + \beta\left[\Theta + S\Theta C\Theta - 2S\Theta + S^3\Theta\right]$$
$$C_z = S\Theta\left(S\Theta - \Theta - S\Theta C\Theta\right) + \beta\left[\left(2 - S^2\Theta\right)\left(1 - C\Theta\right) - \Theta S\Theta\right]$$
$$C_F = \Theta C\Theta - S\Theta\left(1 - \Theta S\Theta\right) + \beta\left[2\left(\Theta - S\Theta\right) - 2S\Theta\left(1 - C\Theta\right) - S\Theta + \Theta C\Theta + \Theta S^2\Theta\right]$$
$$C_w = \left(2 - \Theta S\Theta\right)\left(1 - C\Theta\right) - S^2\Theta + \beta\left[\Theta S\Theta\left(1 + C\Theta\right) - \Theta^2 + 4\left(1 - C\Theta\right) - 3S^2\Theta\right]$$
$$C_{N_2} = S\phi_{N_2} - \phi_{N_2}C\phi_{N_2} - \phi_{N_2}S\phi_{N_2}S\Theta + \beta\left[3S\phi_{N_2} - 2\phi_{N_2} - \phi_{N_2}C\phi_{N_2} + S\Theta\left[2\left(1 - C\phi_{N_2}\right) - \phi_{N_2}S\phi_{N_2}\right]\right]$$

After substituting equation (1.2) we get

$$\delta = \frac{M_{xo}R^2}{2EI}C_x + \frac{M_{zo}R^2}{2EI}C_z + \frac{wR^4}{2EI}\left(C_w + \Theta C_F\right) + \frac{N_1R^3}{4EI}C_F + \frac{N_2R^3}{2EI}\left(C_{N_2} + C_F\right) \tag{4.8}$$

Finally, after substituting for M_{xo} from equation (2.1) and for M_{zo} from equation (3.5) we get

$$\delta = \delta_w + \delta_{N_1} + \sum_{i=1}^{n}\delta_{N_2}(i), \quad \text{if } iVT_h > \frac{1}{2}L \quad \text{then } n = i - 1. \tag{4.9}$$

in which

$$\begin{aligned}
\delta_w &= \frac{wR^4}{2EI}\left[C_w + \Theta C_F + (S\Theta - \Theta C\Theta)C_x + C_{Mw}C_z\right] \\
\delta_{N_1} &= \frac{N_1R^3}{4EI}\left[C_F + (1 - C\Theta)C_x + C_{MN_1}C_z\right] \\
\delta_{N_2}(i) &= \frac{N_2R^3}{2EI}\left[C_{N_2} + C_F + \left(C\phi_{N_2} - C\Theta\right)C_x + C_{MN_2}C_z\right]
\end{aligned} \tag{4.10}$$

For small Θ the coefficients reduce to

$$C_{Mw} = \frac{1}{3}\Theta^2, \quad C_{MN_1} = \frac{1}{2}\Theta, \quad C_{MN_2} = \frac{1}{2}\Theta\left(1 - 4\alpha^2\right), \quad \alpha = \frac{VT_h i}{L}$$

$$\begin{aligned}
C_x &= \Theta - \Theta + \frac{2}{3}\Theta^3 - \Theta^3 + \beta\left[\Theta + \Theta - \frac{2}{3}\Theta^3 - 2\Theta + \frac{1}{2}\Theta^3 + \Theta^3\right] \\
&= -\frac{1}{3}\Theta^3 + \frac{5}{6}\beta\Theta^3
\end{aligned}$$

$$\begin{aligned}
C_z &= \Theta\left(\Theta - \frac{1}{6}\Theta^3 - \Theta - \Theta + \frac{2}{3}\Theta^3\right) + \beta\left[\left(2 - \Theta^2\right)\left(\frac{1}{2}\Theta^2 - \frac{1}{24}\Theta^4\right) - \Theta^2 + \frac{1}{6}\Theta^4\right] \\
&= -\Theta^2 - \frac{5}{12}\beta\Theta^4
\end{aligned}$$

$$C_F = \Theta - \frac{1}{2}\Theta^3 - \Theta + \frac{1}{6}\Theta^3 + \Theta^3$$

$$+\beta\left[\frac{1}{3}\Theta^3 - \frac{1}{60}\Theta^5 - \Theta^3 + \frac{1}{4}\Theta^5 + \frac{1}{6}\Theta^3 - \frac{1}{60}\Theta^5 - \frac{1}{2}\Theta^3 + \frac{1}{24}\Theta^5 + \Theta^3 - \frac{1}{3}\Theta^5\right]$$

$$= \frac{2}{3}\Theta^3 - \frac{1}{15}\beta\Theta^5$$

$$C_w = \Theta^2 - \frac{7}{12}\Theta^4 - \Theta^2 + \frac{1}{3}\Theta^4$$

$$+\Theta^2\beta\left[2 - \frac{5}{6}\Theta^2 + \frac{17}{120}\Theta^4 - 1 + 2 - \frac{1}{6}\Theta^2 + \frac{1}{180}\Theta^4 - 3 + \Theta^2 - \frac{2}{15}\Theta^4\right]$$

$$= -\frac{1}{4}\Theta^4 + \frac{1}{72}\beta\Theta^6$$

$$C_{N_2} = \phi_{N_2} - \frac{1}{6}\phi_{N_2}^3 - \phi_{N_2} + \frac{1}{2}\phi_{N_2}^3 - \phi_{N_2}^2\Theta$$

$$+\phi_{N_2}\beta\left[3 - \frac{1}{2}\phi_{N_2}^2 + \frac{1}{40}\phi_{N_2}^4 - 2 - 1 + \frac{1}{2}\phi_{N_2}^2 - \frac{1}{24}\phi_{N_2}^4 + S\Theta\left(-\frac{1}{12}\phi_{N_2}^3 + \frac{1}{6}\phi_{N_2}^3\right)\right]$$

$$= -\phi_{N_2}^2\Theta\left(1 - \frac{1}{3}\frac{\phi_{N_2}}{\Theta}\right) + \beta\left(-\frac{1}{60}\phi_{N_2}^5 + \frac{1}{12}\phi_{N_2}^4\Theta\right) \to -4\alpha^2\Theta^3\left(1 - \frac{2}{3}\alpha\right)$$

$$\delta_w \to \frac{wR^4}{2EI}\left[-\frac{1}{4}\Theta^4 + \frac{2}{3}\Theta^4 + Order\Theta^6 - \frac{1}{3}\Theta^4\right] = \frac{w(R\Theta)^4}{12EI} = \frac{wL^4}{384EI} \tag{4.11}$$

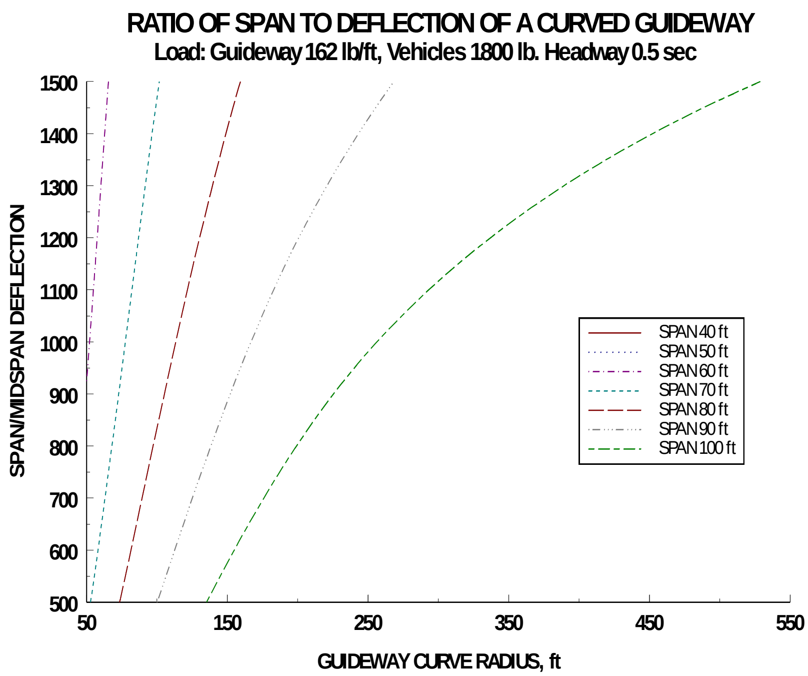
$$\delta_{N_1} \to \frac{N_1R^3}{4EI}\left[\frac{2}{3}\Theta^3 + \frac{1}{2}\Theta^2 \cdot Order\Theta^3 - \frac{1}{2}\Theta^3\right] = \frac{N_1(R\Theta)^3}{24EI} = \frac{N_1L^3}{192EI} \tag{4.12}$$

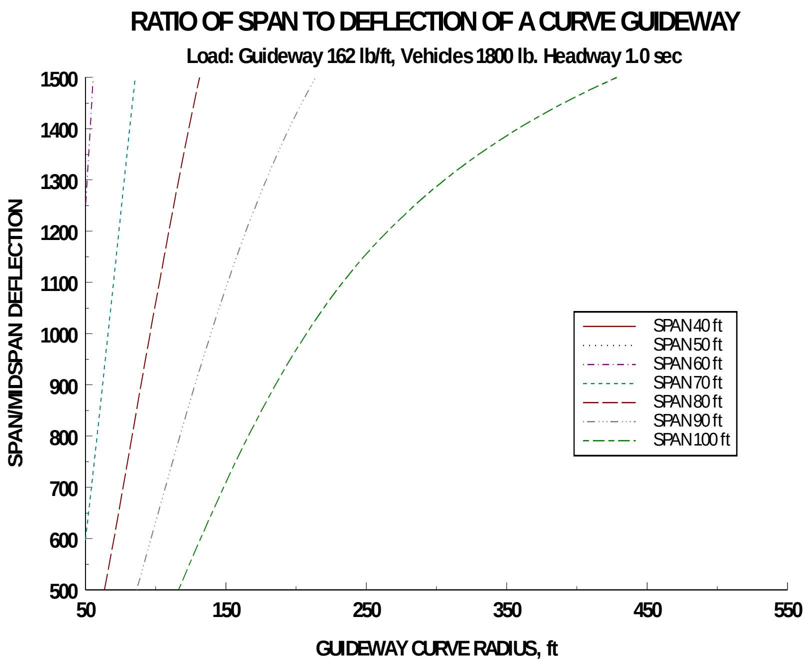
$$\begin{aligned}\delta_{N_2}(i) &\to \frac{N_2R^3}{2EI}\begin{bmatrix} -4\alpha^2\Theta^3\left(1 - \frac{2}{3}\alpha\right) + \frac{2}{3}\Theta^3 \\ -\frac{1}{6}\Theta^5\left(1 - 4\alpha^2\right)\left(1 - \frac{5}{2}\beta\right) - \frac{1}{2}\Theta^3\left(1 - 4\alpha^2\right)\end{bmatrix} \\ &= \frac{N_2L^3}{16EI}\left(\frac{1}{6} - 4\alpha^2 + \frac{8}{3}\alpha^3 + 2\alpha^2\right) \\ &= \frac{N_2L^3}{96EI}\left(1 - 12\alpha^2 + 16\alpha^3\right) = \frac{N_2L^3}{96EI}\left[1 - 4\alpha(3 - 4\alpha)\right]\end{aligned} \tag{4.13}$$

All three of the limit deflections agree with the solution for a straight beam, which is given by equation (6.5), and for the third term by equation (7.13).

5. The Numerical Solution

Equation (4.9) is solved numerically by means of the program given following the following two graph, the first calculated for vehicle spacing corresponding to 0.5 sec headway and the second for 1 sec headway. All of the parameters used are given near the beginning of the program. Cases for which the ratio of span to deflection much above 1000 are overdesigned, and cases for which the span/deflection ratio is much under 1000 are under designed. This conclusion follows a recent people-mover specification that requires a span/deflection ratio of at least 1000. On the other hand, AASHTO specifies a deflection no more than 1/800, so there is some design flexibility. Note that with half-second headway, 90-ft spans in curves can be used if the curve radius is above about 140, which with a horizontal comfort acceleration given by equation (1.3) corresponds to a cruising speed of 25 mph or 41 km/hr or higher. If 90-ft spans are to be used with curve radii corresponding to lower speeds, the curved guideway section must be supported at the center of the span. Such configurations were modeled in the 1991 Chicago RTA PRT design study.






```
'This program CURVGDWY.BAS calculates the properties of a curved
guideway
DEFDBL A-Z
DIM SoverD(0 TO 6) AS DOUBLE
'INPUT PARAMETERS
dpr = 45 / ATN(1)                'degrees per radian
AcclGrav = 32.174                'acceleration of gravity,
ft/sec^2
Modulus = 29500000               'modulus of elasticity of steel,
psi
PoissonRatio = .287
ShearMod = .5 * Modulus / (1 + PoissonRatio)   'psi
BendMomOfI = 4930                'Guideway vertical moment of in-
ertia, in^4
PolarMoment = .8 * 41.38         'Guideway polar moment of iner-
tia, in^4
GdwyWgt = 162                    'guideway weight, lb/ft
GdwyWgt = GdwyWgt / 12           'guideway weight, lb/in
VehWgt = 1800                    'fully loaded vehicle weight, lb
Headway = .5#                    'time headway, sec
BankAngle = 6 / dpr              'radians
'DERIVED PARAMETERS
TwoEI = 2 * Modulus * BendMomOfI   'lb-in^2
Beta = (Modulus / ShearMod) * BendMomOfI / PolarMoment  '383.3
N1 = VehWgt
N2 = VehWgt
```

```
HorzAccl = AcclGrav * (TAN(BankAngle) + .2 / COS(BankAngle))
'ft/sec^2

CLS
OPEN "CURVGDWY.ASC" FOR OUTPUT AS #1
FOR R = 50 TO 550 STEP 4        'guideway radius, ft, 125 points
   Speed = SQR(R * HorzAccl)    'ft/sec
   Dveh = Speed * Headway * 12  'minimum vehicle spacing, in
   Radius = 12 * R              'guideway radius, in


   'LOAD COEFFICIENTS
   DcoefW = GdwyWgt * Radius ^ 4 / TwoEI      'in
   DcoefN1 = .5 * N1 * Radius ^ 3 / TwoEI     'in
   DcoefN2 = N2 * Radius ^ 3 / TwoEI          'in
   j = 0
   FOR L = 40 TO 100 STEP 10     'span, ft, 7 points
       Span = 12 * L            'span, in

       Theta = .5 * Span / Radius    'radians
       Sth = SIN(Theta)
       Cth = COS(Theta)

       'Calculate the dimensionless coefficients
       Denominator = (1 + Beta) * Theta + (1 - Beta) * Sth * Cth
       Term = 2 * (Beta + 1) + (Beta - 1) * Cth ^ 2
       CMwNum = (1 + Beta) * Theta ^ 2 * Sth - Term * (Sth -
Theta * Cth)
       CMw = CMwNum / Denominator
       Term = Theta * Sth - (1 + Cth ^ 2) * (1 - Cth)
       CMN1Num = Sth * (Theta - Sth * (1 - Cth)) + Beta * Term
       CMN1 = CMN1Num / Denominator

       Term = Theta + Sth * Cth + Sth ^ 3 - 2 * Sth
       Cx = Theta - Sth * Cth - Sth ^ 3 + Beta * Term
       Term = (2 - Sth ^ 2) * (1 - Cth) - Theta * Sth
       Cz = Sth * (Sth - Theta - Sth * Cth) + Beta * Term
       Term = -Sth + Theta * (Cth + Sth ^ 2)
       Term = 2 * (Theta - Sth) - 2 * Sth * (1 - Cth) + Term
       CF = Theta * Cth - Sth * (1 - Theta * Sth) + Beta * Term
       Term = Theta * Sth * (1 + Cth) - Theta ^ 2 + 4 * (1 -
Cth) - 3 * Sth ^ 2
       Cw = (2 - Theta * Sth) * (1 - Cth) - Sth ^ 2 + Beta *
Term

       'Guideway deflection due to guideway weight, in
```

```
       Term = Cw + Theta * CF + (Sth - Theta * Cth) * Cx + CMw *
Cz
       Defl.w = DcoefW * Term

       'Guideway deflection due to load N1 at middle of guide-
way, in
       Term = CF + (1 - Cth) * Cx + CMN1 * Cz
       Defl.N1 = DcoefN1 * Term

       'Deflection due to multiple values of N2, in
       Defl.N2 = 0
       FOR i = 1 TO 10          '10 is higher than will be
reached
          Dv = i * Dveh         'distance from center of span to
vehicle i
          IF Dv > Span / 2 THEN EXIT FOR 'include only vehicles
on span
          Phi = Dv / Radius     'angle of vehicle i from center
of span
          CPhi = COS(Phi)
          SPhi = SIN(Phi)
          Term1 = (Beta + 1) * (Theta * Sth - Phi * SPhi)
          Term2 = Beta * (1 + Cth ^ 2) + Sth ^ 2
          CMN2 = (Term1 - Term2 * (CPhi - Cth)) / Denominator

          Term = 2 * (1 - CPhi) - Phi * SPhi
          Term = 3 * SPhi - 2 * Phi - Phi * CPhi + Sth * Term
          CN2 = SPhi - Phi * (CPhi + SPhi * Sth) + Beta * Term
          Term = CF + CN2 + (CPhi - Cth) * Cx + CMN2 * Cz
          Defl.N2 = Defl.N2 + Term 'sum all vehicles on span
       NEXT i
       Defl.N2 = DcoefN2 * Defl.N2      'deflection, in

       'Total deflection of guideway, in
       Deflection = Defl.w + Defl.N1 + Defl.N2
       SoverD(j) = Span / Deflection 'dimensionless
       PRINT USING "######"; R; L; i;
       PRINT USING "#####.#"; Speed * 60 / 88;
       PRINT USING "####.###"; Defl.w; Defl.N1; Defl.N2; Defl.N1
+ Defl.N2;
       PRINT USING "######.#"; SoverD(j)
       'IF L > 69 AND L < 71 THEN SLEEP
       j = j + 1
   NEXT L
   WRITE #1, R, SoverD(0), SoverD(1), SoverD(2), SoverD(3),
SoverD(4), SoverD(5), SoverD(6)
   'SLEEP
```

```
NEXT R
CLOSE #1
```

6. Deflection of a Straight Beam by Castigliano's Method

Consider a straight beam of length L clamped at both ends and under a uniform load w, a point load N_1 at the center and a pair of loads N_2 spaced a distance D to the left and right of the center load, making the problem symmetric about the midpoint. The upward reaction force at each end of the beam is thus $\frac{1}{2}(wL + N_1) + N_2$. If the coordinate along the beam, x, is measured from the left end of the beam, the moment in the beam is

$$M(x) = M_o - \left[\frac{1}{2}\left(wL + N_1\right) + N_2\right]x + \frac{1}{2}wx^2 + N_2\left\langle x - L/2 + D\right\rangle \tag{6.1}$$

in which M_o is the statically indeterminate moment at the ends of the beam.

The strain energy in the beam is

$$U = 2\int_0^{L/2} \frac{M^2}{2EI}dx \tag{6.2}$$

M_o is determined by the principal of least work, which states that

$$\frac{\partial U}{\partial M_o} = 0 = \frac{2}{EI}\int_0^{L/2} M\,\frac{\partial M}{\partial M_o}\,dx = \frac{2}{EI}\int_0^{L/2} M\,dx \tag{6.3}$$

Substituting equation (6.1)

$$\begin{aligned}
0 &= \int_0^{L/2}\left\{M_o - \left[\frac{1}{2}\left(wL + N_1\right) + N_2\right]x + \frac{1}{2}wx^2 + N_2\left\langle x - L/2 + D\right\rangle\right\}dx \\
&= M_o\frac{L}{2} - \left[\frac{1}{2}\left(wL + N_1\right) + N_2\right]\frac{L^2}{8} + \frac{1}{48}wL^3 + N_2\int_0^D u\,du \\
\therefore M_o &= \frac{2}{L}\left[\frac{wL^3}{24} + \frac{N_1L^2}{16} + \frac{N_2L^2}{8} - N_2\frac{D^2}{2}\right] = \frac{wL^2}{12} + \frac{N_1L}{8} + \frac{N_2L}{4}\left(1 - \frac{4D^2}{L^2}\right)
\end{aligned} \tag{6.4}$$

By Castigliano's Theorem the deflection at the center of the beam is

$$\delta = \frac{\partial U}{\partial N_1} = \frac{2}{EI}\int_0^{L/2} M\,\frac{\partial M}{\partial N_1}\,dx$$

$$= \frac{2}{EI}\int_0^{L/2}\left\{M_o - \left[\frac{1}{2}\left(wL+N_1\right)+N_2\right]x + \frac{1}{2}wx^2 + N_2\left\langle x-L/2+D\right\rangle\right\}\left(-\frac{x}{2}\right)dx$$

$$= -\frac{1}{EI}\left\{M_o\frac{L^2}{8} - \left[\frac{1}{2}\left(wL+N_1\right)+N_2\right]\frac{L^3}{24} + \frac{wL^4}{128} + N_2\int_0^D u\left(u+\frac{L}{2}-D\right)du\right\}$$

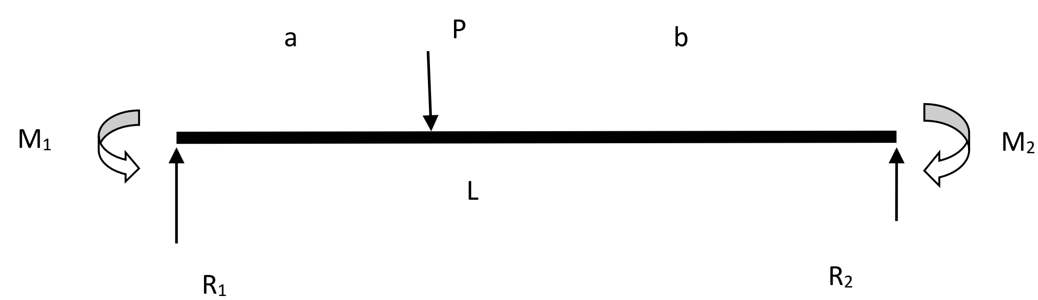
$$= -\frac{1}{EI}\left\{\begin{matrix}\left[\dfrac{wL^2}{12}+\dfrac{N_1L}{8}+\dfrac{N_2L}{4}\left(1-\dfrac{4D^2}{L^2}\right)\right]\dfrac{L^2}{8} \\ -\left[\dfrac{1}{2}\left(wL+N_1\right)+N_2\right]\dfrac{L^3}{24}+\dfrac{wL^4}{128}+N_2\left[\dfrac{D^3}{3}+\left(\dfrac{L}{2}-D\right)\dfrac{D^2}{2}\right]\end{matrix}\right\}$$

$$= \frac{1}{EI}\left\{wL^4\left(-\frac{1}{96}+\frac{1}{48}-\frac{1}{128}\right)+N_1L^3\left(-\frac{1}{64}+\frac{1}{48}\right)+\frac{N_2L^3}{8}\left(-\frac{1}{4}+\frac{1}{3}\right)+\frac{N_2LD^2}{8}+\frac{N_2D^2}{6}\left(D-\frac{3L}{2}\right)\right\}$$

$$= \frac{wL^4}{384EI}+\frac{N_1L^3}{192EI}+\frac{N_2L^3}{96EI}\left[1-4\frac{D^2}{L^2}\left(3-4\frac{D}{L}\right)\right] \tag{6.5}$$

which is a well-known result.

7. <u>Deflection of a Clamped Beam by under a Point Load by Integration</u>



Consider a beam of length L clamped at both ends and subject to a point load P at a distance a from the left end. Thus $b = L - a$. From statics we balance the forces in the vertical direction and the moments about one end — we take the left end. Then we find the following two equations for the four unknowns:

$$R_1 + R_2 = P \tag{7.1}$$

$$M_1 + R_2L = M_2 + Pa \tag{7.2}$$

Since the problem is statically indeterminate we find the necessary additional two equations by solving for the slope and deflection of the beam. Thus

$$
\begin{aligned}
&EIy'' = M(x) = M_1 - R_1 x + P\langle x-a\rangle \qquad \text{if } Q \le 0 \text{ then } \langle Q\rangle = 0. \\
&EIy' = M_1 x - R_1\frac{x^2}{2} + P\frac{\langle x-a\rangle^2}{2} \\
&EIy = M_1\frac{x^2}{2} - R_1\frac{x^3}{6} + P\frac{\langle x-a\rangle^3}{6}
\end{aligned} \tag{7.3}
$$

Since $y'(L)=0$ we have

$$
M_1 = R_1\frac{L}{2} - P\frac{b^2}{2L} \tag{7.4}
$$

Since $y(L)=0$ we have

$$
M_1 = R_1\frac{L}{3} - P\frac{b^3}{3L^2} \tag{7.5}
$$

Equating (7.4) and (7.5) we have

$$
\begin{aligned}
&R_1\frac{L}{2} - P\frac{b^2}{2L} = R_1\frac{L}{3} - P\frac{b^3}{3L^2} \\
&R_1 = 6P\frac{b^2}{L^2}\left(\frac{1}{2} - \frac{b}{3L}\right) = P\frac{b^2}{L^3}(3a+3b-2b) = P\frac{b^2(3a+b)}{L^3}
\end{aligned} \tag{7.6}
$$

Substituting into equation (7.1) we get

$$
\begin{aligned}
R_2 &= P - P\frac{b^2(3a+b)}{L^3} = \frac{P}{L^3}\left(a^3+3a^2b+3ab^2+b^3-3ab^2-b^3\right) \\
&= P\frac{a^2(3b+a)}{L^3}
\end{aligned} \tag{7.7}
$$

Substituting equation (7.6) into equation (7.5) we get

$$
M_1 = P\frac{b^2(3a+b)}{L^3}\frac{L}{2} - P\frac{b^2}{2L} = P\frac{b^2}{2L}\left(\frac{(3a+b)}{L} - \frac{(a+b)}{L}\right) = P\frac{ab^2}{L^2} \tag{7.8}
$$

This is the moment at the left end, called M_{zo} in Section 3. But there we applied a pair of loads N_1, one a distance D_v to the left of the center point, and another a distance D_v to the right of the center point. In this case, the moment at the left (or right) end would be

$$\begin{aligned} M_{end} &= P\frac{ab^2}{L^2}+P\frac{ba^2}{L^2}=\frac{P}{L^2}ab\left(b+a\right)=\frac{P}{L^2}\left(\frac{L}{2}-D_v\right)\left(\frac{L}{2}+D_v\right)\left(\frac{L}{2}+D_v+\frac{L}{2}-D_v\right) \\ &=\frac{P}{L^2}\frac{L^2}{4}\left(1-\frac{4D_v^2}{L^2}\right)L=\frac{PL}{4}\left(1-\frac{4D_v^2}{L^2}\right) \end{aligned} \tag{7.9}$$

which, in the N_2 term, agrees with equations (3.7) and (6.4).

Substituting equations (7.7) and (7.8) into equation (7.2) we get

$$\begin{aligned} M_2 &= M_1+R_2L-Pa=P\frac{ab^2}{L^2}+P\frac{a^2\left(3b+a\right)}{L^2}-Pa \\ &=\frac{Pa}{L^2}\left[b^2+a\left(3b+a\right)-a^2-2ab-b^2\right]=P\frac{ba^2}{L^2} \end{aligned} \tag{7.10}$$

Now let $a = L/2 - D$ where $D > 0$. Then, the deflection at x = L/2, from the third of equations (7.3) is

$$\begin{aligned} EIy &= M_1\frac{x^2}{2}-R_1\frac{x^3}{6}+P\frac{\langle x-a\rangle^3}{6} \\ EIy\left(\frac{L}{2}\right) &= P\frac{ab^2}{L^2}\frac{L^2}{8}-P\frac{b^2\left(3a+b\right)}{L^3}\frac{L^3}{48}+\frac{P}{48}\left(b-a\right)^3 \\ &=\frac{P}{48}\left[6ab^2-3ab^2-b^3+b^3-3b^2a+3ba^2-a^3\right] \\ y\left(\frac{L}{2}\right) &= \frac{Pa^2\left(3b-a\right)}{48EI}=\frac{P}{48EI}\left(\frac{L}{2}-D\right)^2\left(3\frac{L}{2}+3D-\frac{L}{2}+D\right) \\ &=\frac{PL^3}{192EI}\left(1-2\frac{D}{L}\right)^2\left(1+4\frac{D}{L}\right) \end{aligned} \tag{7.11}$$

With $D = 0$ we get the well-known formula for the deflection of a clamped beam under a concentrated load at the center. Now let $a = L/2 + D$ where $D > 0$. Then from the third of equations (7.3) the deflection at the center is

$$EIy\left(\frac{L}{2}\right) = M_1\frac{x^2}{2} - R_1\frac{x^3}{6} = P\frac{ab^2}{L^2}\frac{L^2}{8} - P\frac{b^2\left(3a+b\right)}{L^3}\frac{L^3}{48}$$

$$= \frac{P}{48}\left(6ab^2 - 3ab^2 - b^3\right) = \frac{Pb^2\left(3a-b\right)}{48}$$

$$= \frac{P}{48}\left(\frac{L}{2} - D\right)^2\left(3\frac{L}{2} + 3D - \frac{L}{2} + D\right) = \frac{P}{48}\left(\frac{L}{2} - D\right)^2\left(L + 4D\right) \tag{7.12}$$

$$y\left(\frac{L}{2}\right) = \frac{PL^3}{192EI}\left(1 - 2\frac{D}{L}\right)^2\left(1 + 4\frac{D}{L}\right)$$

which is the same as equation (7.11) as must be expected. With a pair of point load at a distance $x = \frac{L}{2} \pm D$ the center deflection is

$$y\left(\frac{L}{2}\right) = \frac{PL^3}{96EI}\left(1 - 2\frac{D}{L}\right)^2\left(1 + 4\frac{D}{L}\right) \tag{7.12}$$

Note that

$$\left(1 - 2\frac{D}{L}\right)^2\left(1 + 4\frac{D}{L}\right) = \left(1 - 4\frac{D}{L} + 4\frac{D^2}{L^2}\right)\left(1 + 4\frac{D}{L}\right)$$

$$= 1 - 4\frac{D}{L} + 4\frac{D^2}{L^2} + 4\frac{D}{L}\left(1 - 4\frac{D}{L} + 4\frac{D^2}{L^2}\right)$$

$$= 1 - 12\frac{D^2}{L^2} + 16\frac{D^3}{L^3} = 1 - 4\frac{D^2}{L^2}\left(3 - 4\frac{D}{L}\right) = 0 \quad \text{if} \quad \frac{D}{L} = \frac{1}{2}$$

Thus, equation (9.12) can be written in the form

$$y\left(\frac{L}{2}\right) = \frac{PL^3}{96EI}\left[1 + 4\frac{D^2}{L^2}\left(4\frac{D}{L} - 3\right)\right] \tag{7.13}$$

which with $P = N_2$ agrees with equation (6.5).

The Critical Speed

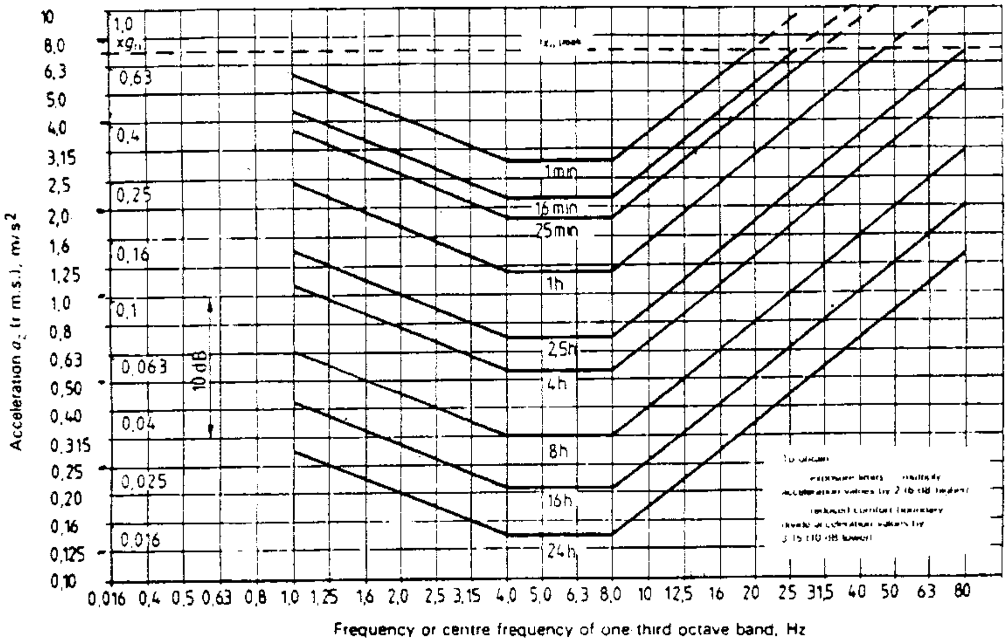


FIGURE 3-1 SPECTRAL COMPOSITION OF ACCELERATION

The above figure is taken from the International Standards Organization maximum oscillatory acceleration as a function of frequency that will be comfortable to the passengers riding in a vehicle. The frequency of motion is given by

$$f = \frac{V}{L} \tag{1}$$

in which V is the speed of travel over a flexible guideway and L is the distance between support posts. The vertical displacement felt by a passenger is

$$y(t) = \frac{1}{2}\Delta_{\max}\sin(2\pi ft) \tag{2}$$

Where $\Delta_{\max}$ is the mid-span deflection of the guideway from the posts and t is time. Differentiating twice with respect to t, the maximum vertical acceleration is

$$a_{\max} = 2\pi^2 f^2 \Delta_{\max} \tag{3}$$

The above chart gives the root-mean-square (rms) value of acceleration, which is less by the square root of 2. Thus

$$a_{rms} = \sqrt{2}\pi^2 f^2 \Delta_{\max} \tag{4}$$

If and the post spacing is $L = 90$ ft, $V = 90\text{ ft/sec} = 61.4\text{ mph}$, and if $f = 4\text{ Hz}$ the speed is $V = 360\text{ ft/sec} = 245\text{ mph}$. Critical speeds of interest will be between these two values, for which the comfort acceleration is linearly decreasing in the above log-log plot. Thus

$$\begin{aligned} &\ln(a_{rms}) = \ln\left(a_{rms_1}\right) + m\left[\ln(f) - \ln(1)\right] \\ &\text{in which } m = \frac{\left[\ln\left(a_{rms_4}\right) - \ln\left(a_{rms_1}\right)\right]}{\left[\ln(4) - \ln(1)\right]} \end{aligned} \tag{5}$$

Although $\ln(1) = 0$ we keep this form to clarify units in the final equation.

From the above chart, let us use the curve for 1-hour duration of the acceleration. Then we see from the chart that $a_{rms_1} = 2.4\text{ m/s}^2$ and $a_{rms_4} = 1.2\text{ m/s}^2$. Thus

$$m = \frac{\ln\left(\dfrac{a_{rms_4}}{a_{rms_1}}\right)}{\ln(4)} = -\frac{\ln(2)}{2\ln(2)} = -\frac{1}{2} \tag{6}$$

From the properties of logarithms, equation (5) becomes

$$a_{rms} = e^{\left[\ln\left(a_{rms_1}\right) + m\ln\left(\frac{f}{1}\right)\right]} = a_{rms_1} e^{m\ln\left(\frac{f}{1}\right)} = a_{rms_1}\left(\frac{f}{1\text{Hz}}\right)^m \tag{7}$$

From the middle form of equations (7) we see that if we set $f = 4\text{ Hz}$ that $a_{rms} = a_{rms_4}$. By substituting equation (7) into equation (4) we obtain

$$a_{rms_1}\left(\frac{f}{1}\right)^m = \sqrt{2}\pi^2\left(\frac{f}{1}\right)^2 (1\,\mathrm{Hz})^2\,\Delta_{\max}$$

$$\left(\frac{f}{1}\right)^{2-m} = \frac{a_{rms_1}}{\sqrt{2}\pi^2\Delta_{\max}(1\,\mathrm{Hz})^2} \tag{8}$$

$$\frac{f}{1\,\mathrm{Hz}} = \frac{V_{cr}}{(1\,\mathrm{Hz})L} = \left[\frac{a_{rms_1}}{\sqrt{2}\pi^2\Delta_{\max}(1\,\mathrm{Hz})^2}\right]^{\frac{1}{2-m}}$$

For the lightest tube stringer $\Delta_{\max} = 0.925\,\mathrm{in}\times\frac{1\,\mathrm{ft}}{12\,\mathrm{in}}\times 0.3048\,\frac{\mathrm{m}}{\mathrm{ft}} = 0.0235\,\mathrm{m}$ and for the largest tube stringer $\Delta_{\max} = 0.734\,\mathrm{in}\times\frac{1\,\mathrm{m}}{39.37\,\mathrm{in}} = 0.0186\,\mathrm{m}.$ Thus

$$V_{cr} = \frac{L}{s}\left[\frac{2.4\,\mathrm{m/s^2}}{\sqrt{2}\pi^2 0.0236\,\mathrm{m/s^2}}\right]^{0.4} = 2.21L/\mathrm{s} = 2.21(90\,\mathrm{ft/s}) = 199\,\mathrm{ft/s} = 136\,\mathrm{mph},$$

and for the largest stringer, $V_{cr} = 149\,\mathrm{mph}.$

The Polar Moment of Inertia of the ITNS Guideway

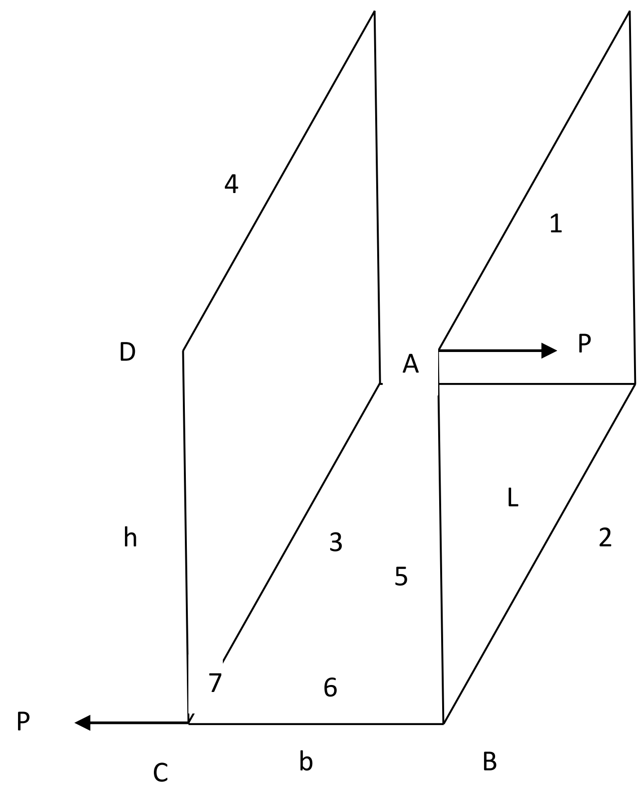


Figure B-1. A Segment of the Guideway

Figure B-1 is intended to be a three-dimensional view of an idealized model of a segment of our guideway, meaning that it is a segment between a pair of U-frames space a distance L apart. The 7 members of the segment are depicted as lines representing their centroids. Realizing that the eye can visualize an object intended to be three-dimensional in two ways, I intend that the U-frame lower and to the left is in the foreground. I assume in this analysis that the U-frame in the background is fixed but that the four stringers labeled 1, 2, 3, 4 are continuous beams, meaning that, from Appendix C the deflection at the end of each of these four beams due to a side load P is

$$\Delta = \frac{PL^3}{1.608\,EI} \tag{1}$$

Any contribution of the diagonals to torsional rigidity will be very small because of their orientation and small stiffness in bending. The moments of inertia $I_1 = I_4$ and $I_2 = I_3$. The three elements of the U-frame in the foreground are labeled 5, 6, and 7. For them $I_5 = I_6 = I_7$. They are of height h and width b. The four corners of the U-frame are labeled A, B, C, and D. The U-frame in the foreground is subjected to a pair of equal and opposite forces of magnitude P producing a clockwise twisting moment on the guideway of magnitude Ph. The object of this analysis is to determine the relationship between the twisting moment and the angle of twist.

To solve the problem, we separate the elements and the corners into a series of 11 free-body diagrams. Thus

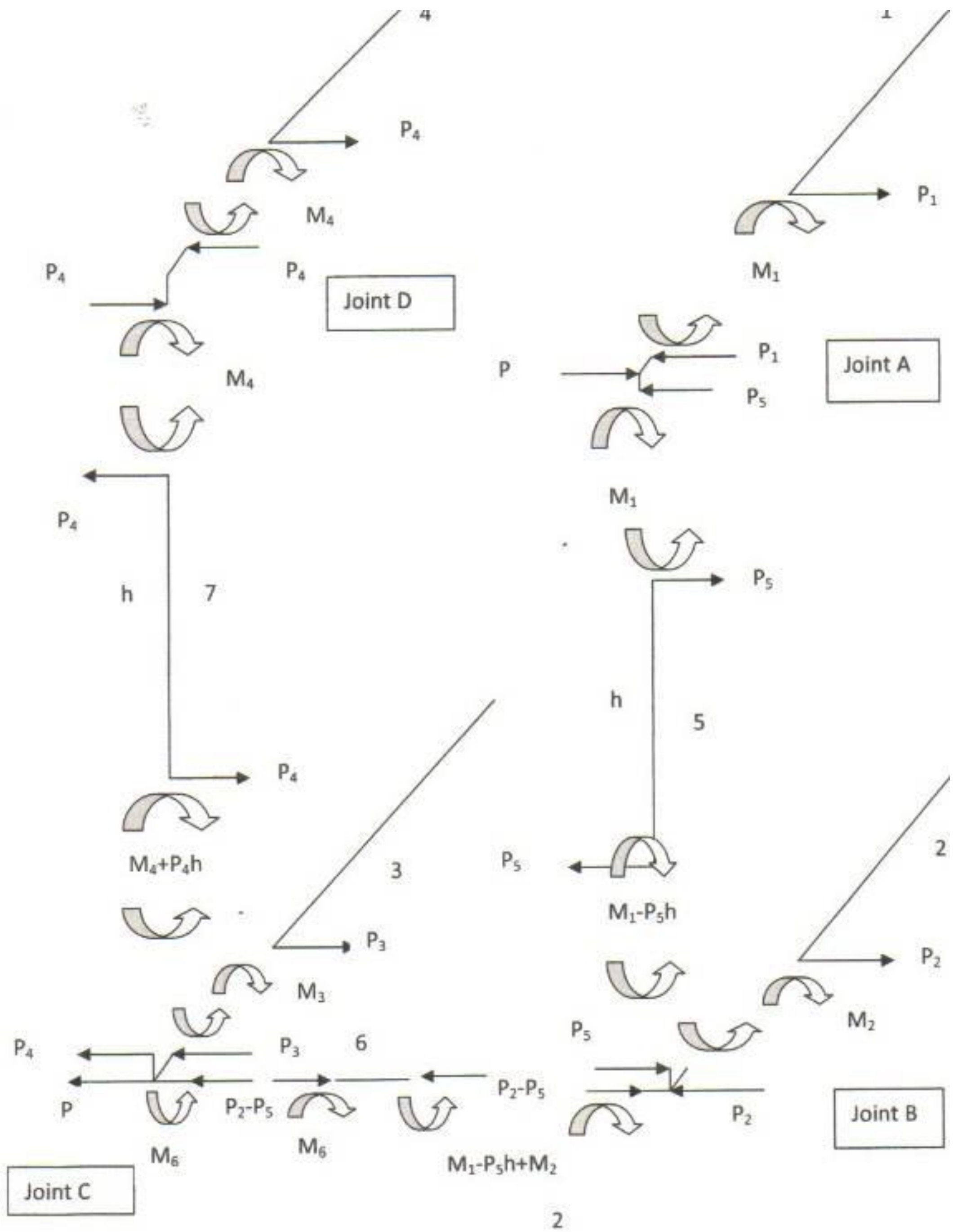


Figure B-2. The Moment Diagram for a Section of Guideway.

From the free-body diagrams shown in Figure B-2, we have the following equations of statics,

$$\begin{aligned}
&P = P_1 + P_5 = -P_2 - P_3 - P_4 + P_5 \\
&\therefore\ P_5 = P - P_1 \\
&\quad P_1 + P_2 + P_3 + P_4 = 0 \\
&M_6 = M_1 + M_2 - P_5 h = -M_3 - M_4 - P_4 h \\
&\therefore M_1 + M_2 + M_3 + M_4 = \left(P - P_1 - P_4\right) h
\end{aligned} \tag{2}$$

which can be verified by considering the whole U-frame as a free-body diagram. Assume the deflections of the four joints are positive to the right and denoted by $\Delta_a, \Delta_b, \Delta_c, \Delta_d$. While twist will cause some vertical displacement, it does not enter into this analysis. Using equation (1)

$$\begin{aligned}
\Delta_a &= \frac{P_1 L^3}{1.608\, EI_1} \\
\Delta_b &= \frac{P_2 L^3}{1.608\, EI_2} \\
\Delta_c &= \frac{P_3 L^3}{1.608\, EI_2} \\
\Delta_d &= \frac{P_4 L^3}{1.608\, EI_1}
\end{aligned} \tag{3}$$

in which the moments of inertia *I* of the members 1 and 4 are less than the moment of inertia of the members 2 and 3. The moments of inertia are the sums of the values for the stringers and the running surfaces. Let the twist angles of each of the four stringers, positive clockwise, be denoted by

$$\theta_1, \theta_2, \theta_3, \theta_4$$

Then, these angles are given in terms of the end moments by the equations

$$\begin{aligned}
\theta_1 &= \frac{M_1 L}{GI_{p1}} \\
\theta_2 &= \frac{M_2 L}{GI_{p2}} \\
\theta_3 &= \frac{M_3 L}{GI_{p2}} \\
\theta_4 &= \frac{M_4 L}{GI_{p1}}
\end{aligned} \tag{4}$$

in which L is the distance between U-frames, G is the shear modulus, and I_p is the sum of the polar moments of inertia of the stringer and running surface.

Consider the three elements of the U-frame, which are numbered 5, 6, and 7. Each is subject to both an end load and an end moment. Thus, if the distance along the beam element is x, beginning at x = 0 and ending at x = l, the equations for the moment, slope, and deflection are

$$\begin{aligned} EIy'' &= M + P(l-x) \\ EIy' &= EIy'(0) + Mx + P\left(lx - \frac{x^2}{2}\right) \\ EIy &= EIy(0) + EIy'(0)x + M\frac{x^2}{2} + P\left(l\frac{x^2}{2} - \frac{x^3}{6}\right) \end{aligned} \tag{5}$$

Thus, the slope and deflection at $x = l$ are

$$\begin{aligned} y'(l) &= y'(0) + \frac{Ml}{EI} + \frac{Pl^2}{2EI} \\ y(l) &= y(0) + y'(0)l + \frac{Ml^2}{2EI} + \frac{Pl^3}{3EI} \end{aligned} \tag{6}$$

Apply these equations to the vertical U-frame elements 5 and 7. Thus

$$\begin{aligned} \theta_1 &= \theta_2 - \frac{M_1 h}{EI_5} + \frac{P_5 h^2}{2EI_5} \\ \Delta_a &= \Delta_b + \theta_2 h - \frac{M_1 h^2}{2EI_5} + \frac{P_5 h^3}{3EI_5} \\ \theta_4 &= \theta_3 - \frac{M_4 h}{EI_5} - \frac{P_4 h^2}{2EI_5} \\ \Delta_d &= \Delta_c + \theta_3 h - \frac{M_4 h^2}{2EI_5} - \frac{P_4 h^3}{3EI_5} \end{aligned} \tag{7}$$

For element #6

$$\begin{aligned} \theta_2 &= \theta_3 + \left(M_1 - P_5 h + M_2\right)\frac{b}{EI_5} \\ \Delta_c &= \Delta_b \end{aligned} \tag{8}$$

There is an equation for the vertical displacements of joints B and C, but it is not of interest here. Now substitute equations (3) and (4) into equations (7) and (8), using equations (2), to get

$$
\begin{aligned}
&\frac{M_1 L}{GI_{p1}} = \frac{M_2 L}{GI_{p2}} - \frac{M_1 h}{EI_5} + \frac{(P-P_1)h^2}{2EI_5} \\
&\frac{P_1 L^3}{1.608\,EI_1} = \frac{P_2 L^3}{1.608\,EI_2} + \frac{M_2 L h}{GI_{p2}} - \frac{M_1 h^2}{2EI_5} + \frac{(P-P_1)h^3}{3EI_5} \\
&\frac{M_4 L}{GI_{p1}} = \frac{M_3 L}{GI_{p2}} - \frac{M_4 h}{EI_5} - \frac{P_4 h^2}{2EI_5} \\
&\frac{P_4 L^3}{1.608\,EI_1} = \frac{P_3 L^3}{1.608\,EI_2} + \frac{M_3 L h}{GI_{p2}} - \frac{M_4 h^2}{2EI_5} - \frac{P_4 h^3}{3EI_5} \\
&\frac{M_2 L}{GI_{p2}} = \frac{M_3 L}{GI_{p2}} + \left(M_1 - Ph + P_1 h + M_2\right)\frac{b}{EI_5} \\
&P_3 = P_2 \\
&P_1 + P_2 + P_3 + P_4 = 0 \\
&M_1 + M_2 + M_3 + M_4 + P_1 h + P_4 h = Ph
\end{aligned}
\tag{9}
$$

Equations (9) form a set of 8 equations in 8 unknowns. They can be arranged in the following form. In so doing the second and fourth equations are multiplied by the ratio I_{p1}/I_{p2}.

$$
\begin{aligned}
&\left(1 + \frac{h}{L}\frac{G}{E}\frac{I_{p1}}{I_5}\right)M_1 - \frac{I_{p1}}{I_{p2}}M_2 + \frac{1}{2}\frac{h}{L}\frac{G}{E}\frac{I_{p1}}{I_5}P_1 h = \frac{1}{2}\frac{h}{L}\frac{G}{E}\frac{I_{p1}}{I_5}Ph \\
&\frac{1}{2}\frac{h}{L}\frac{G}{E}\frac{I_{p1}}{I_5}M_1 - \frac{I_{p1}}{I_{p2}}M_2 + \left(\frac{1}{1.608}\frac{L^2}{h^2}\frac{G}{E}\frac{I_{p1}}{I_1} + \frac{1}{3}\frac{h}{L}\frac{G}{E}\frac{I_{p1}}{I_5}\right)P_1 h - \frac{1}{1.608}\frac{L^2}{h^2}\frac{G}{E}\frac{I_{p1}}{I_2}P_2 h = \frac{1}{3}\frac{h}{L}\frac{G}{E}\frac{I_{p1}}{I_5}Ph \\
&\left(1 + \frac{h}{L}\frac{G}{E}\frac{I_{p1}}{I_5}\right)M_4 - \frac{I_{p1}}{I_{p2}}M_3 + \frac{1}{2}\frac{h}{L}\frac{G}{E}\frac{I_{p1}}{I_5}P_4 h = 0 \\
&\frac{1}{2}\frac{h}{L}\frac{G}{E}\frac{I_{p1}}{I_5}M_4 - \frac{I_{p1}}{I_{p2}}M_3 + \left(\frac{1}{1.608}\frac{L^2}{h^2}\frac{G}{E}\frac{I_{p1}}{I_1} + \frac{1}{3}\frac{h}{L}\frac{G}{E}\frac{I_{p1}}{I_5}\right)P_4 h - \frac{1}{1.608}\frac{L^2}{h^2}\frac{G}{E}\frac{I_{p1}}{I_2}P_3 h = 0 \\
&M_1 + \left(1 - \frac{E}{G}\frac{L}{b}\frac{I_5}{I_{p2}}\right)M_2 + \frac{E}{G}\frac{L}{b}\frac{I_5}{I_{p2}}M_3 + P_1 h = Ph
\end{aligned}
\tag{10}
$$

Let

$$
m_1 = \frac{h}{L}\frac{G}{E}\frac{I_{p1}}{I_5},\ m_2 = \frac{I_{p1}}{I_{p2}},\ m_3 = \frac{1}{1.608}\frac{L^2}{h^2}\frac{G}{E}\frac{I_{p1}}{I_1},\ m_4 = \frac{E}{G}\frac{L}{b}\frac{I_5}{I_{p2}}
\tag{11}
$$

Using the notation of equations (11) equations (10) are simplified to

$$
\begin{aligned}
&\left(1+m_1\right)M_1 - m_2 M_2 + \tfrac{1}{2} m_1 P_1 h = \tfrac{1}{2} m_1 P h \\
&\tfrac{1}{2} m_1 M_1 - m_2 M_2 + \left(m_3 + \tfrac{1}{3} m_1\right) P_1 h - m_3 \frac{I_1}{I_2} P_2 h = \tfrac{1}{3} m_1 P h \\
&\left(1+m_1\right)M_4 - m_2 M_3 + \tfrac{1}{2} m_1 P_4 h = 0 \\
&\tfrac{1}{2} m_1 M_4 - m_2 M_3 + \left(m_3 + \tfrac{1}{3} m_1\right) P_4 h - m_3 \frac{I_1}{I_2} P_2 h = 0 \\
&M_1 + \left(1-m_4\right)M_2 + m_4 M_3 + P_1 h = P h
\end{aligned} \tag{12}
$$

in which we have substituted the sixth of equations (9) into the fourth of equations (10). We can solving the first and second of equations (12) for M_1 and M_2 and the third and fourth for M_4 and M_3. Note that the solution of the third and fourth of equations (12) are obtained by substituting subscript 4 for 1, the subscript 3 for 2, and by setting $P = 0$. The solutions can be written in the form

$$
\begin{aligned}
M_1 &= a_{11} P_1 h - a_{12} P_2 h + a_1 P h \\
M_2 &= a_{21} P_1 h - a_{22} P_2 h - a_2 P h \\
M_4 &= a_{11} P_4 h - a_{12} P_2 h \\
M_3 &= a_{21} P_4 h - a_{22} P_2 h
\end{aligned} \tag{13}
$$

in which

$$
\begin{aligned}
&a_{11} = \frac{1}{3}\left(\frac{6m_3 - m_1}{m_1 + 2}\right), \quad a_{12} = \left(\frac{2m_3}{m_1+2}\right)\frac{I_1}{I_2}, \quad a_1 = \frac{1}{3}\left(\frac{m_1}{m_1+2}\right) \\
&a_{21} = \frac{12 m_3\left(m_1+1\right) + m_1\left(m_1+4\right)}{6 m_2\left(m_1+2\right)}, \quad a_{22} = \frac{2m_3\left(m_1+1\right)}{m_2\left(m_1+2\right)}\frac{I_1}{I_2}, \quad a_2 = \frac{m_1\left(m_1+4\right)}{6m_2\left(m_1+2\right)}
\end{aligned} \tag{14}
$$

Substitute the moments from equation (13) into the fifth of equations (12). The result is

$$
\left[a_{11} + \left(1-m_4\right)a_{21} + 1\right]P_1 - \left(a_{12} + a_{22}\right)P_2 + m_4 a_{21} P_4 = \left[1 - a_1 + \left(1-m_4\right)a_2\right]P \tag{15}
$$

From the seventh and eighth of equations (9)

$$
P_2 = -\tfrac{1}{2}\left(P_1 + P_4\right) \tag{16}
$$

Substitute equation (16) into equation (15). The result can be written in the form

$$c_{11}P_1 + c_{14}P_4 = c_1 P \tag{17}$$

in which

$$\begin{aligned} c_{11} &= a_{11} + (1-m_4)a_{21} + 1 + \tfrac{1}{2}(a_{12}+a_{22}) \\ c_{14} &= m_4 a_{21} + \tfrac{1}{2}(a_{12}+a_{22}) \\ c_1 &= 1 - a_1 + (1-m_4)a_2 \end{aligned} \tag{18}$$

Substitute the moment equations (13) into the last of equations (9). The result, after substituting equation (16), can be written in the form

$$c_{21}(P_1 + P_4) = c_2 P \tag{19}$$

in which

$$c_{21} = 1 + a_{11} + a_{21} + a_{12} + a_{22}, \quad c_2 = 1 - a_1 + a_2 \tag{20}$$

The solution of simultaneous equations (17) and (19) is

$$\begin{aligned} P_1 &= \left[\frac{c_1 c_{21} - c_2 c_{14}}{c_{21}(c_{11} - c_{14})} \right] P \\ P_4 &= \left[\frac{c_{11} c_2 - c_{21} c_1}{c_{21}(c_{11} - c_{14})} \right] P \end{aligned} \tag{21}$$

By substituting equations (21), the sixth of equations (9), and equation (16) into equations (3) we have the sidewise displacements of the four joints of Figure C-1. From them the twist angle of a guideway segment of length L is

$$\Theta = \frac{\Delta_a + \Delta_d - 2\Delta_c}{2h}. \tag{22}$$

<u>The Torsional Stiffness of the Guideway</u>

From equations (4) the polar moment of inertia for the entire guideway is

$$I_p = \frac{M}{G\Theta / L} = \frac{(P/G)h}{\Theta / L} = \frac{2h^2 L}{G\left(\Delta_a / P + \Delta_d / P - 2\Delta_c / P\right)} \tag{23}$$

With the upper force in Figure B-1 to the right and the lower force to the left, we can expect the deflections of the joints A and D to be positive, but the deflection of the joints C and D to be negative. Therefore, there will be a point at a distance *a* above the joints C and D at which the sidewise deflection will be zero. From simple geometry,

$$\frac{\frac{1}{2}(\Delta_a+\Delta_d)}{h-a}=\frac{-\Delta_c}{a} \quad \text{or} \quad \tfrac{1}{2}\left(\Delta_a+\Delta_d\right)a=\left(a-h\right)\Delta_c$$

or

$$a=h\left(\frac{-\Delta_c}{\frac{1}{2}\left(\Delta_a+\Delta_d\right)+\Delta_c}\right) \tag{24}$$

<u>The Numerical Solution</u>

For steel $E=29.5(10)^6$ psi.

$$\frac{G}{E}=\frac{1}{2\left(1+\mu\right)},\ \ \mu=\text{Poisson's Ratio}=0.30 \text{ for steel}$$

From Section 3 of the main paper

$$L=54",\ h=31",\ b=28"$$

For the tubular stringers, the bending and polar moments of inertia about the neutral axis are

$$I=\frac{\pi}{64}\left[OD^4-\left(OD-t\right)^4\right],\ I_p=2I,\ OD=4"=\text{outside diameter},\ t=\text{wall thickness}$$

For the 4x4x1/4” upper angle running surfaces, the moments of inertia for sidewise deflection is, from the *Manual of Steel Construction*, 3.00 in^4. For the 8x6x1/2” lower angle running surfaces 44.4 in$^{4.}$

The centroids of the upper angle running surface and the tubular stringer are at the same height in the guideway, but the centroid of the lower angle is 4.472” above the centroid of the lower tubular stringers. Since the distance between the upper and lower tubular stringers is 31”, we will diminish the moment of inertia of the lower angles by the factor (31-4.472)/31 = 0.856. Therefore, we take the effective moment of inertia of the lower angles as 0.856(44.4) = 38.0 in^4. The moment of inertia of the channel cross-sections of the U-frames is 8.89 in^4.

The polar moment of inertia of an angle is[19]

[19] S. Timoshenko, *Strength of Materials, Part II*, 2nd Ed., D. van Nostrand Company, Inc. 1941.

$$I_p = \frac{1}{3}(s_1 + s_2 - t)t^3$$

in which s_1 and s_2 are the two sides and t is the thickness. Thus, for the upper running surfaces

$$I_p = \frac{1}{3}(4+4-1/4)\left(\frac{1}{4}\right)^3 = 0.0404\,\text{in}^4$$

For the lower running surfaces

$$I_p = \frac{1}{3}(8+6-1/2)\left(\frac{1}{2}\right)^3 = 0.5625\,\text{in}^4.$$

Results obtained in an Excel Spreadsheet follow:

Solution for Twist of the Guideway

Modulus of Elasticity of Steel =	29,500,000	29,500,000	29,500,000	29,500,000	psi
Poisson's ratio for Steel =	0.30	0.30	0.30	0.30	
E/G =	2.60	2.60	2.60	2.60	
Shear Modulus for Steel =	11,346,154	11,346,154	11,346,154	11,346,154	psi
L =	54	54	54	54	in
h =	26	26	26	26	in
b =	28	28	28	28	in
Stringer OD =	4	4	4	4.5	in
Stringer wall thickness =	0.174	0.233	0.315	0.315	in
Stringer moment of inertia =	2.048	2.682	3.515	5.071	in^4
Stringer polar moment of inertia =	4.096	5.364	7.030	10.143	in^4
Moment of inertia of upper angles =	3.00	3.00	3.00	3.00	in^4
Effective moment of inertia of lower angles =	38.00	38.00	38.00	38.00	in^4
Moment of inertia of U-frame, I5 =	8.89	8.89	8.89	8.89	in^4
I1 =	5.05	5.68	6.51	8.07	in^4
I2 =	40.05	40.68	41.51	43.07	in^4
Polar moment of upper running surfaces =	0.0404	0.0404	0.0404	0.0404	in^4

Polar moment of lower running surfaces =	0.5625	0.5625	0.5625	0.5625	in^4
Ip1 =	4.1363	5.4042	7.0701	10.1832	in^4
Ip2 =	4.6584	5.9263	7.5923	10.7054	in^4
m1 =	0.0862	0.1126	0.1473	0.2121	
m2 =	0.8879	0.9119	0.9312	0.9512	
m3 =	0.8454	0.9813	1.1197	1.3017	
m4 =	9.5691	7.5218	5.8714	4.1640	
a11 =	0.7967	0.9113	1.0200	1.1449	
a12 =	0.1022	0.1298	0.1637	0.2205	
a1 =	0.0138	0.0178	0.0229	0.0320	
a21 =	1.0232	1.1735	1.3358	1.5705	
a22 =	0.2400	0.3006	0.3774	0.5129	
a2 =	0.0317	0.0401	0.0509	0.0708	
c11 =	-6.7996	-5.5272	-4.2165	-2.4572	
c14 =	9.9617	9.0424	8.1133	6.9061	
c1 =	0.7148	0.7210	0.7291	0.7441	
c21 =	3.1621	3.5152	3.8969	4.4488	
c2 =	1.0179	1.0223	1.0280	1.0388	
Denominator =	-53.0011	-51.2152	-48.0474	-41.6557	
P1/P =	0.1487	0.1310	0.1145	0.0928	
P4/P =	0.1732	0.1598	0.1494	0.1408	
P2/P =	-0.1610	-0.1454	-0.1319	-0.1168	
P3/P =	-0.1610	-0.1454	-0.1319	-0.1168	

L^3/1.608/E =	0.003320	0.003320	0.003320	0.003320	in^5/lb
DeltaA/P =	0.0000978	0.0000765	0.0000583	0.0000381	In/lb
DeltaD/P =	0.0001139	0.0000934	0.0000761	0.0000579	in/lb
DeltaB/P =	-0.0000133	-0.0000119	-0.0000105	-0.0000090	in/lb
Twist angle per lb =	0.00000458	0.00000372	0.00000299	0.00000219	radians/lb
Polar Moment of Inertia of Guideway =	26.99	33.23	41.38	56.43	in^4
a/h =	0.1442	0.1623	0.1861	0.2306	
M1/Ph =	0.1487	0.1560	0.1612		
M2/Ph =	0.1591	0.1574	0.1518		
M3/Ph =	0.1545	0.1645	0.1739		
M4/Ph =	0.2159	0.2313	0.2493		
P =	709	709	709		lb
M1 =	2741	2876	2972		in-lb
M2 =	2932	2902	2798		in-lb
M3 =	2847	3032	3206		in-lb
M4 =	3980	4263	4595		in-lb
Shear Stress1 =	1338	1072	845		psi
Shear Stress2 =	1432	1082	796		psi
Shear Stress3 =	1390	1131	912		psi
Shear Stress4 =	1943	1590	1307		psi

Ride Comfort over Slope Discontinuity

Abstract

The acceleration experienced by a passenger riding in an ITNS vehicle is derived and presented in graphical form. The problem required the solution of four second-order nonlinear differential equations for vertical motion of the vehicle, pitch motion of the vehicle, forward motion of the vehicle and vertical motion of the passenger. These differential equations are solved by a numerical technique, which is presented. A time step of 0.00001 seconds was used which is practical because double-precision numbers are used. The results show that with a slope discontinuity of half of a degree, the passenger acceleration will be around one eighth of a *g*, somewhat greater or less depending on the seat stiffness. Seat stiffness of about 200 lb/in is about right. The damping ratio should be in the range of 0.3 to 0.5. Higher tire stiffness does not increase passenger acceleration significantly. A negative slope change does not change the amplitude of the passenger acceleration. Moving the passenger back increases the passenger acceleration by a small amount.

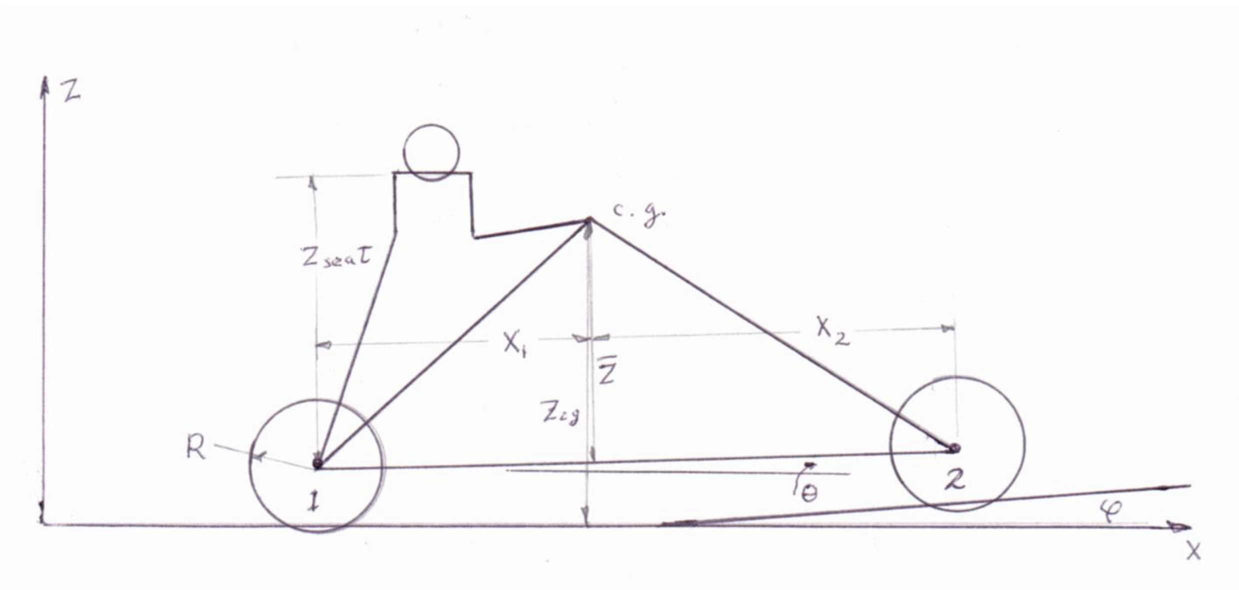


Figure 1. Vehicle engaging a Slope Discontinuity.

Introduction

The problem addressed is to determine the maximum vertical acceleration of a passenger in an ITNS vehicle as the vehicle passes over a slope discontinuity and therefore the maximum slope discontinuity that can be permitted. Figure 1 gives the essential features of the vehicle.

Notation and Geometry

- Motion is referred to an orthogonal reference frame *x*-*z*, where *x* is horizontal and *z* vertical.
- The rear wheel is called wheel #1 and the front wheel is called wheel #2.
- The radius of each undeflected tire is *R*.
- An upward dynamic force is applied to the two rear tires $F_1 = 2k\delta_1 + 2c\dot{\delta}_1$ where k is the tire stiffness is and c is the tire damping coefficient. δ_1 is the tire deflection and $\dot{\delta}_1$ is the time rate of the deflection. The dot indicates the time derivative.
- Similarly, the upward force on the two front tires is $F_2 = 2k\delta_2 + 2c\dot{\delta}_2$.
- In body coordinates, in which the *x*-body axis lies along the line connecting the rear and front axles, the center of gravity of the vehicle is a distance $\bar{z}$ above that line.
- The center of gravity of the vehicle of empty weight *W* is marked c. g. and is a distance z_{cg} above the *x*-axis.
- The angle between the x–axis and the x –body axis is the small angle θ.
- The motion of the vehicle is given in terms of $z_{cg}(t)$ and $\theta(t)$.
- The rear-wheel axle is a distance $x_1$behind the vehicle c. g. in body coordinates.
- The front-wheel axle is a distance x_2 in front of the vehicle c. g. in body coordinates.
- The sum $x_1 + x_2 = L$, the wheelbase.
- The rear tire is deflected an amount δ_1 and the front tire is deflected an amount δ_2.
- At time *t* = 0 $R - \delta_1 = z_{cg} - \bar{z} - x_1\theta,\ R - \delta_2 = z_{cg} - \bar{z} + x_2\theta$.
- Thus

$$\delta_1 = R + \bar{z} - z_{cg} + x_1\theta, \qquad \delta_2 = R + \bar{z} - z_{cg} - x_2\theta$$
$$\dot{\delta}_1 = -\dot{z}_{cg} + x_1\dot{\theta}, \qquad \dot{\delta}_2 = -\dot{z}_{cg} - x_2\dot{\theta}$$

- Note that $\delta_1 - \delta_2 = L\theta$ is greater than zero if the rear tire has deflected more than the front tire, which must be the case.

- The undeflected passenger seat is a distance z_{seat} above the rear axle in body coordinates.
- The passenger has deflected the seat by an amount δ_p.
- The passenger is a point mass of weight W_p locate at a vertical distance $z_p < z_{seat}$.
- The downward dynamic force of the passenger on the seat is $F_s = k_s\delta_p + c_s\dot{\delta}_p$ in which $k_s$is the seat stiffness and $c_s$is the seat damping factor.
- The passenger seat is a distance x_p forward of the rear axle.
- The *z*-coordinate of the undeflected passenger seat is
  $z_s = z_{cg} - \bar{z} + z_{seat} - x_{1p}\theta, \quad \dot{z}_s = \dot{z}_{cg} - x_{1p}\dot{\theta}$ where $x_{1p} \equiv x_1 - x_p$.
- The deflection of the seat due to the weight of the passenger is $\delta_p = z_s - z_p$.

- The initial seat deflection is $\delta_p = \frac{W_p}{k_s}$ or $z_p = z_s - \frac{W_p}{k_s}$.
- In Figure 1, the front wheel has passed a step discontinuity φ_{max} in the slope of the running surface.
- As a result of the slope discontinuity, the vehicle tips up the mentioned small angle θ from the *x*-axis.
- Since the front wheel has passed over the slope discontinuity, the force F_2 makes an angle φ with the *z*-axis and hence, since angle φ is very small, there is a component of the force $F_2 sin\varphi \cong F_2\varphi$ opposing the motion of the vehicle.
- The length of the tire contact area is determined by the Pythagorean Theorem: $R^2 = (R-\delta)^2 + (0.5L_{contact})^2$, or $0.5L_{contact} = \sqrt{2R\delta - \delta^2}$. We are interested in the segment of the circumference of the tire between first contact and the half way point to the last contact, which is $C(\delta) = R * ATN\left(\frac{\sqrt{2R\delta-\delta^2}}{R-\delta}\right)$.
- The vehicle moves to the right in Figure 1at the initial speed $\dot{x}(0)$.
- At $t = 0$ the value of x at the front axle is zero just as the front tire reaches the point of the sharp change in slope.
- The tire, having finite stiffness, engages the slope change gradually as it advances up the slope. We can simulate this change by assuming that the grade changes linearly to a maximum when $x = C(\delta)$.
- In the region from $x = 0\ to\ x = C(\delta)$ take the slope to be the continuous curve
$$\varphi(x) = \frac{1}{2}\ \varphi_{max}\left[1 - cos\left(\frac{\pi x}{C(\delta)}\right)\right], \dot{\varphi}(x) = \frac{1}{2}\ \varphi_{max} sin\left(\frac{\pi x}{C(\delta)}\right)\frac{\pi}{C(\delta)}\dot{x}$$
- In the region $x \geq C(\delta)$ $\quad \varphi(x) = \varphi_{max}$.
- The rear tire begins to engage the slope change when $x = L$.
- The function zSlope$(x) = x\varphi(x)$ is the vertical distance between the x-axis and the sloped running surface when $0 \leq x \leq C(\delta)$, and zSlope$(x) =$ xφ_{max} when $x > C(\delta)$.
- The *z*-distance of the front axle from the surface *z* = 0 can be represented in two ways: $R - \delta_2 + \text{zSlope}(x) = z_{cg} - \bar{z} + x_2\theta$. Thus $\delta_2 = R + \bar{z} - z_{cg} - x_2\theta + \text{zSlope}(x)$.
- When $x \leq L$ the *z*-coordinate of the rear axle is a distance $R - \delta_1 = z_{cg} - \bar{z} - x_1\theta$ above the *x*-axis, and when $x > L$ the value $zSlope(x - L)$ is added to $R - \delta_1$.
- In summary
$$\delta_2 = R + \bar{z} - z_{cg} - x_2\theta + \text{zSlope}(x), \qquad \dot{\delta}_2 = -\dot{z}_{cg} - x_2\dot{\theta} + \dot{\text{z}}Slope(x)$$
where $\dot{\text{z}}Slope(x) = \dot{x}\varphi(x) + x\dot{\varphi}(x)\ if\ x < C(\delta)$ or $\varphi_{max}\dot{x}$ if $x \geq C(\delta)$.
$$\delta_1 = R + \bar{z} - z_{cg} + x_1\theta + \beta\text{zSlope}(x - L), \qquad \dot{\delta}_1 = -\dot{z}_{cg} + x_1\dot{\theta} + \beta\dot{\text{z}}Slope(x - L)$$
where $\beta = 0\ if x < L$ and $\beta = 1\ if\ x \geq L$.

Statics

Then, the static balance of forces on a vehicle of empty weight W is

$$F_1 L = W x_2 + W_p(L - x_p)$$
$$F_2 L = W x_1 + W_p x_p$$

In the static situation at t = 0

$$F_1 = 2k\delta_1 = 2k\left(R + \bar{z} - z_{cg_s} + x_1\theta\right)$$
$$F_2 = 2k\delta_2 = 2k\left(R + \bar{z} - z_{cg_s} - x_2\theta\right)$$
$$\frac{F_1 - F_2}{2k} = L\theta_s = \frac{1}{2kL}\left[W x_2 + W_p(L - x_p) - W x_1 - W_p x_p\right]$$
$$\theta_s = \frac{1}{2kL^2}\left[W(x_2 - x_1) + W_p(L - 2x_p)\right]$$

where θ_s is the static and initial value of θ. The static initial value of z_{cg_s} is then

$$z_{cg_s} = R + \bar{z} + x_1\theta_s - \frac{1}{2kL}\left[W x_2 + W_p(L - x_p)\right] =$$
$$R + \bar{z} + \frac{1}{2kL^2}\left\{W(x_2 - x_1)x_1 + W_p(L - 2x_p)x_1 - WLx_2 - W_pL(L - x_p)\right\}$$
$$= R + \bar{z} - \frac{1}{2kL^2}\left[W(x_1^2 + x_2^2) + W_p\left[x_2 L + x_p(x_1 - x_2)\right]\right]$$

Note that if $x_1 = x_2$ and $x_p = L/2$ then, as expected, $\theta_s = 0$ and $z_{cg_s} = R + \bar{z} - \frac{W+W_p}{4k}$. The static balance of forces on the passenger seat is

$$W_p = k_s\left(z_{s_s} - z_p\right), \qquad z_p = z_{s_s} - \frac{W_p}{k_s}$$

where

$$z_{s_s} = z_{cg_s} - \bar{z} + z_{seat} - x_{1p}\theta_s.$$

Dynamics

The dynamic values of the tire forces are

$$F_1 = 2k\delta_1 + 2c\dot{\delta}_1$$
$$F_2 = 2k\delta_2 + 2c\dot{\delta}_2$$

The dynamic value of the upward force of the passenger seat on the passenger is

$$F_s = k_s\delta_p + c_s\dot{\delta}_p$$

Let r_g be the radius of gyration of the empty vehicle. The equations of motion of the vehicle are

$$\frac{W}{g}\ddot{z}_{cg} = -W + F_1 + F_2 - F_s$$
$$\frac{W}{g}r_g^2\ddot{\theta} = F_2x_2 - F_1x_1 + F_sx_{1p}$$
$$\frac{W}{g}\ddot{x} = -\varphi(x)F_2 - \beta\varphi(x-L)F_1$$

The vertical equation of motion of the passenger is

$$\frac{W_p}{g}\ddot{z}_p = -W_p + F_s$$

<u>Numerical Solution of the Four Second-Order Differential Equations</u>

The method use is described in the internal paper "A Practical Method for Numerical Solution of Differential Equations." A general second-order differential equation can be expressed as

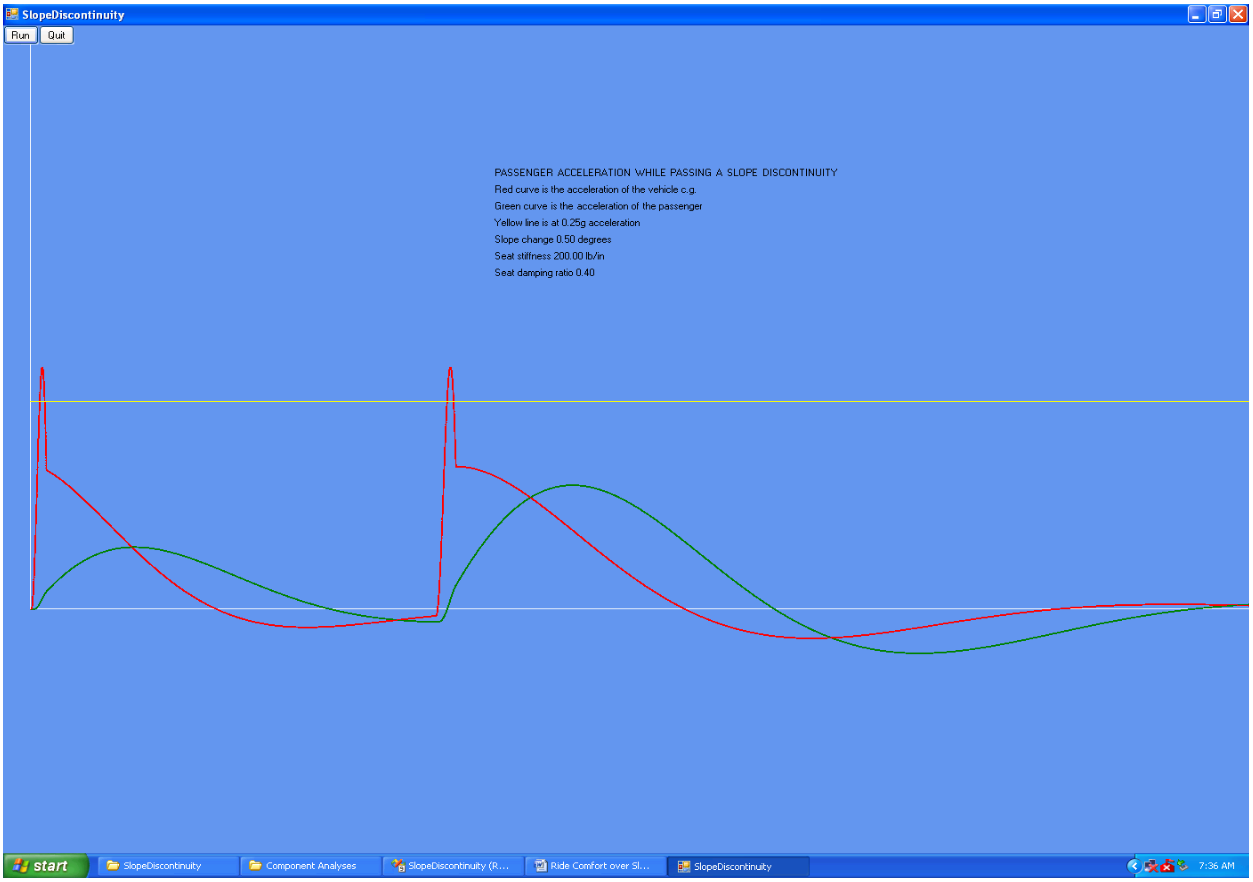
$$\frac{d^2y}{dt^2} = f(y,\dot{y},t)$$

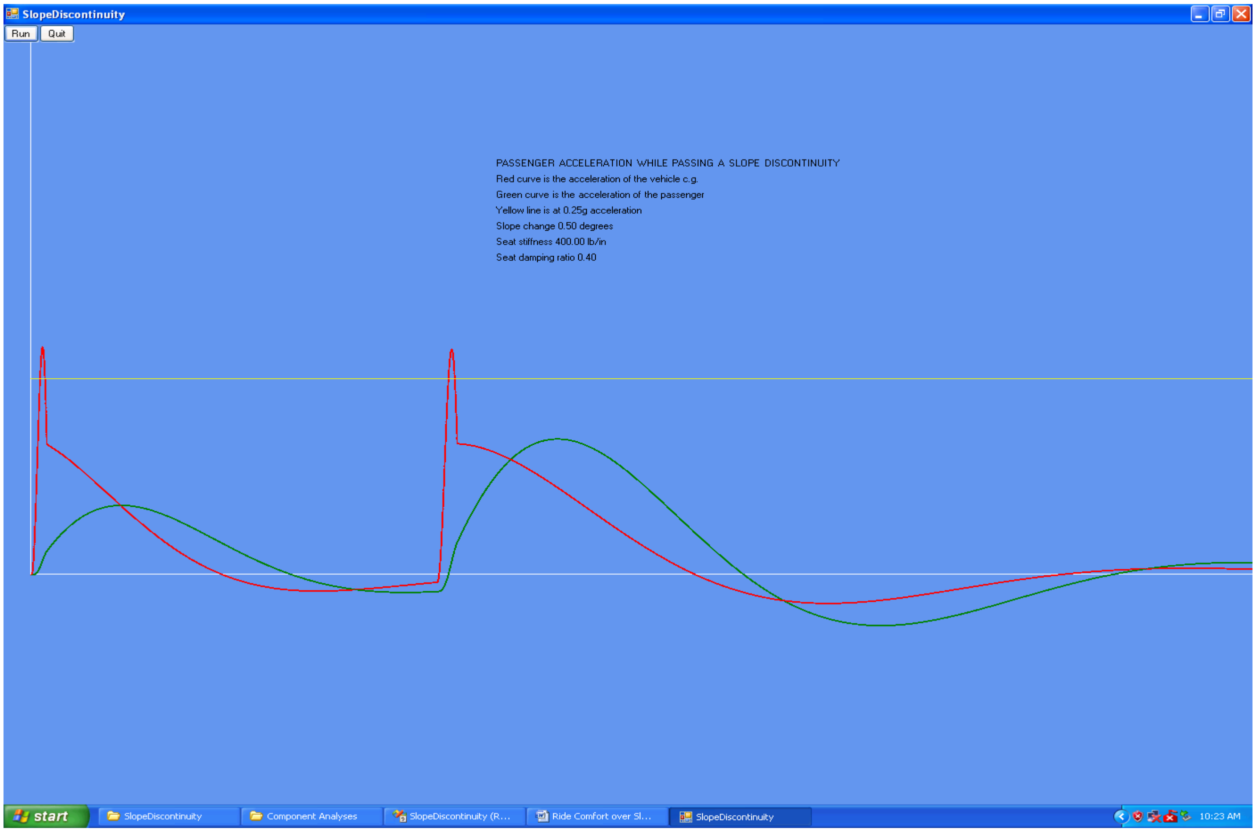
It can be expressed as two first-order differential equations

$$\frac{du}{dt} = f(u,y,t), \qquad \frac{dy}{dt} = u$$

For each time step calculate

$$f_{i-1} = f_i$$
$$f_i = f(t,u,y)$$
$$u_{i+1} = u_i + \frac{1}{2}\Delta t(3f_i - f_{i-1})$$
$$y_{i+1} = y_i + \frac{1}{2}\Delta t(u_{i+1} + u_i)$$
$$u_i = u_{i+1}$$


SlopeDiscontinuity
Run
Quit
PASSENGER ACCELERATION WHILE PASSING A SLOPE DISCONTINUITY
Red curve is the acceleration of the vehicle c.g.
Green curve is the acceleration of the passenger
Yellow line is at 0.25g acceleration
Slope change 0.50 degrees
Seat stiffness 200.00 lb/in
Seat damping ratio 0.40
start
SlopeDiscontinuity
Component Analyses
Ride Comfort over Sl...
7:36 AM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Seat damping ratio 0.40
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Component Analyses
Ride Comfort over Sl...
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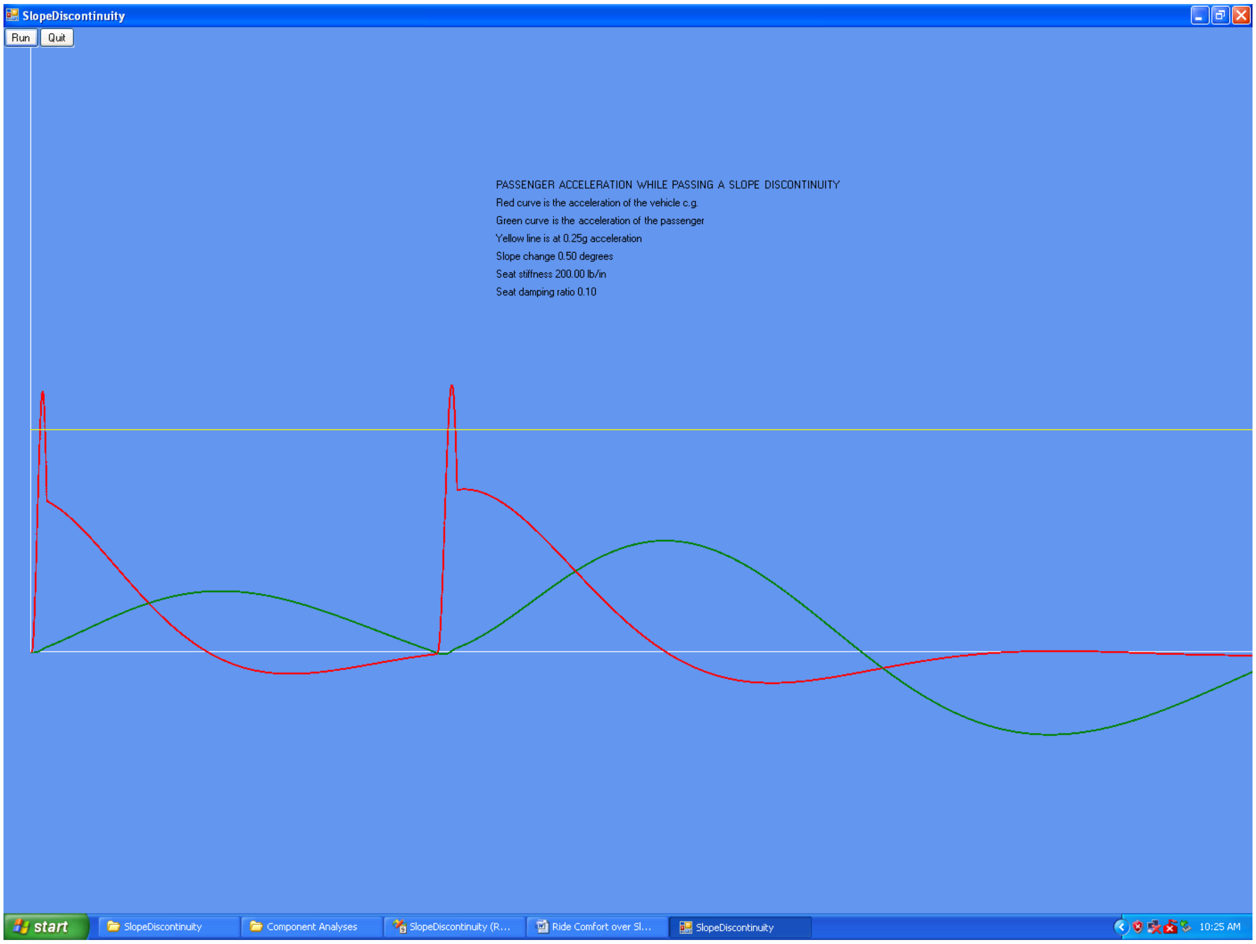


The Program

```
Public Class MainForm
    Public dpr As Double = 180 / Math.PI          'degrees per radian
    Public g As Double = 32.174 * 12              'acceleration of gravity,
in/sec^2
    Public kTire As Double = 800                  'tire stiffness, lb/in
    Public kSeat As Double = 200                  'seat stiffness, lb/in
    Public radiusGyr As Double = 40               'radius of gyration of empty
vehicle, in
    Public Wveh As Double = 1200                  'vehicle empty weight, lb
    Public Wpass As Double = 200                  'passenger weight, lb
    Public OmegaN = Math.Sqrt(2 * kTire * g / Wveh)
    Public OmegaS = Math.Sqrt(kSeat * g / Wpass)
    Public zetaTire As Double = 0.5               'tire damping ratio
    Public zetaSeat As Double = 0.4               'seat damping ratio
    Public cTire = 2 * zetaTire * OmegaN
    Public cSeat = 2 * zetaSeat * OmegaS
    Public Slope As Double = 0.5 / dpr            'sudden change in slope of
running surface, rad
    Public L As Double = 80                       'distance between front and
rear wheel axles, in
    Public x1 As Double = 0.4 * L                 'distance from rear axle to
vehicle c.g., in
```

```
    Public x2 As Double = L - x1                'distance from vehicle c.g.
to front axle, in
    Public xp As Double = 0.3 * L               'distance from rear axle to
passenger c.g., in
    Public x1p As Double = x1 - xp              'distance between passenger
c. g. and vehicle c. g.
    Public radiusTire As Double = 13.25         'tire radius, in
    Public zBar As Double = 36                  'distance from axles to c.g
in body axes, in
    Public RzBar As Double = radiusTire + zBar
    Public zSeat As Double = 48                 'distance from wheel axles to
undeflected seat, in
    Public zSeatzBar As Double = zSeat - zBar

    Public TheRunThread As Threading.Thread
    Dim objGraphics As System.Drawing.Graphics
    Dim ObjFont = New System.Drawing.Font("Arial", 60)


    Private Sub MainForm_Load(ByVal sender As System.Object, ByVal e As
System.EventArgs) Handles MyBase.Load
        Dim radiusgyrSq As Double = radiusGyr ^ 2
        Dim oldcgAccel, oldThetaA, oldPassAccel, oldxDotDot As Double
        Dim NewzcgDot, NewthetaDot, NewxDot, NewzpDot As Double
        Dim Defl1, Defl1Dot, Defl2, Defl2Dot, Deflp, DeflpDot, zs, zsDot As
Double
        Dim c1, c2, Phi1, Phi2, zSlope1, zSlope2, zSlope1Rate, zSlope2Rate As
Double
        Dim F1, F2, Fs As Double

        'initial upward motion of vehicle c.g.
        Dim zcgAccel As Double = 0
        Dim zcgDot As Double = 0
        Dim Term As Double = x2 * L + xp * (x1 - x2)
        Dim zcg As Double = RzBar - (Wveh * (x1 ^ 2 + x2 ^ 2) + Wpass * Term)
/ 2 / kTire / L ^ 2

        'initial counterclockwise angular motion of vehicle
        Dim ThetaAccel As Double = 0
        Dim thetaDot As Double = 0
        Dim theta As Double = (Wveh * (x2 - x1) + Wpass * (L - 2 * xp)) / 2 /
kTire / L ^ 2

        'initial vehicle forward motion
        Dim xDotDot As Double = 0
        Dim xDot As Double = 30 * (88 / 60) * 12 'vehicle forward speed
entering slope change
        Dim x As Double = 0

        'initial passenger motion
        Dim PassAccel As Double = 0
        Dim zpDot As Double = 0
        Dim zp As Double = zcg + zSeatzBar - x1p * theta - Wpass / kSeat

        'deflection of front tire
        Defl1 = RzBar - zcg + x1 * theta
```

```
        c1 = radiusTire * Math.Atan(Math.Sqrt(2 * radiusTire * Defl1 - Defl1
^ 2) / (radiusTire - Defl1))
        Defl2 = RzBar - zcg - x2 * theta
        c2 = radiusTire * Math.Atan(Math.Sqrt(2 * radiusTire * Defl2 - Defl2
^ 2) / (radiusTire - Defl2))

        Dim t As Double = 0
        Dim delT As Double = 0.00001

        Dim xGraph, yGraph As Single
        Dim x0 As Single = 30
        Dim y0 As Single = 700
        Dim tScale As Single = 3000
        Dim aScale As Single = 1000

        Dim HorizontalLocation = 550
        Dim VerticalLocation = 150

        objGraphics = Me.CreateGraphics
        objGraphics.DrawLine(Pens.White, x0, y0, x0, 0)
        objGraphics.DrawLine(Pens.White, x0, y0, 2000, y0)

        Do
            c1 = radiusTire * Math.Atan(Math.Sqrt(2 * radiusTire * Defl1 -
Defl1 ^ 2) / (radiusTire - Defl1))
            If x < L Then
                Phi1 = 0
                zSlope1 = 0
                zSlope1Rate = 0
            ElseIf x < L + c1 Then
                Phi1 = 0.5 * Slope * (1 - Math.Cos(Math.PI * (x - L) / c1))
                zSlope1 = Phi1 * (x - L)
                zSlope1Rate = Phi1 * xDot + (x - L) * 0.5 * Slope *
Math.Sin(Math.PI * (x - L) / c1) * Math.PI * xDot / c1
            Else
                Phi1 = Slope
                zSlope1 = Slope * (x - L)
                zSlope1Rate = Slope * xDot
            End If

            If x < c2 Then
                Phi2 = 0.5 * Slope * (1 - Math.Cos(Math.PI * x / c2))
                zSlope2 = Phi2 * x
                zSlope2Rate = Phi2 * xDot + x * 0.5 * Slope *
Math.Sin(Math.PI * x / c2) * Math.PI * xDot / c2
            Else
                Phi2 = Slope
                zSlope2 = Slope * x
                zSlope2Rate = Slope * xDot
            End If

            'Deflections and deflection rates
            Defl1 = RzBar - zcg + x1 * theta + zSlope1
            Defl1Dot = -zcgDot + x1 * thetaDot + zSlope1Rate
            Defl2 = RzBar - zcg - x2 * theta + zSlope2
            Defl2Dot = -zcgDot - x2 * thetaDot + zSlope2Rate
            zs = zcg + zSeatzBar - x1p * theta
```

```
            zsDot = zcgDot - x1p * thetaDot
            Deflp = zs - zp
            DeflpDot = zsDot - zpDot

            'Forces
            F1 = 2 * kTire * Defl1 + 2 * cTire * Defl1Dot
            F2 = 2 * kTire * Defl2 + 2 * cTire * Defl2Dot
            Fs = kSeat * Deflp + cSeat * DeflpDot

            'Solution of the difference equations
            oldcgAccel = zcgAccel
            zcgAccel = -g + (F1 + F2 - Fs) * g / Wveh
            NewzcgDot = zcgDot + 0.5 * delT * (3 * zcgAccel - oldcgAccel)
            zcg = zcg + 0.5 * delT * (NewzcgDot + zcgDot)
            zcgDot = NewzcgDot

            oldThetaA = ThetaAccel
            ThetaAccel = (F2 * x2 - F1 * x1 + Fs * x1p) * g / Wveh /
radiusgyrSq
            NewthetaDot = thetaDot + 0.5 * delT * (3 * ThetaAccel -
oldThetaA)
            theta = theta + 0.5 * delT * (NewthetaDot + thetaDot)
            thetaDot = NewthetaDot

            oldxDotDot = xDotDot
            xDotDot = -Phi2 * F2 - Phi1 * F1
            NewxDot = xDot + 0.5 * delT * (3 * xDotDot - oldxDotDot)
            x = x + 0.5 * delT * (NewxDot + xDot)
            xDot = NewxDot

            oldPassAccel = PassAccel
            PassAccel = -g + Fs * g / Wpass
            NewzpDot = zpDot + 0.5 * delT * (3 * PassAccel - oldPassAccel)
            zp = zp + 0.5 * delT * (NewzpDot + zpDot)
            zpDot = NewzpDot

            xGraph = x0 + tScale * t
            yGraph = y0 - aScale * PassAccel / g
            objGraphics.FillEllipse(Brushes.Green, xGraph, yGraph, 2, 2)
            yGraph = y0 - aScale * zcgAccel / g
            objGraphics.FillEllipse(Brushes.Red, xGraph, yGraph, 2, 2)
            yGraph = y0 - aScale * 0.25
            objGraphics.FillEllipse(Brushes.Yellow, xGraph, yGraph, 2, 2)

            Application.DoEvents()
            t = t + delT
        Loop Until t > 0.5
        objGraphics.DrawString("PASSENGER ACCELERATION WHILE PASSING A SLOPE
DISCONTINUITY", Me.Font, System.Drawing.Brushes.Black, HorizontalLocation,
VerticalLocation + 20)
        objGraphics.DrawString("Red curve is the acceleration of the vehicle
c.g.", Me.Font, System.Drawing.Brushes.Black, HorizontalLocation,
VerticalLocation + 40)
        objGraphics.DrawString("Green curve is the acceleration of the
passenger", Me.Font, System.Drawing.Brushes.Black, HorizontalLocation,
VerticalLocation + 60)
```

```
        objGraphics.DrawString("Yellow line is at 0.25g acceleration",
Me.Font, System.Drawing.Brushes.Black, HorizontalLocation, VerticalLocation +
80)
        objGraphics.DrawString("Slope change " & FormatNumber(Slope * dpr, 2)
& " degrees", Me.Font, System.Drawing.Brushes.Black, HorizontalLocation,
VerticalLocation + 100)
        objGraphics.DrawString("Seat stiffness " & FormatNumber(kSeat, 2) & "
lb/in", Me.Font, System.Drawing.Brushes.Black, HorizontalLocation,
VerticalLocation + 120)
        objGraphics.DrawString("Seat damping ratio " & FormatNumber(zetaSeat,
2) & " ", Me.Font, System.Drawing.Brushes.Black, HorizontalLocation,
VerticalLocation + 140)

        objGraphics.Dispose()
    End Sub

    Private Sub btnRun_Click(ByVal sender As System.Object, ByVal e As
System.EventArgs) Handles btnRun.Click

    End Sub

    Private Sub btnQuit_Click(ByVal sender As System.Object, ByVal e As
System.EventArgs) Handles btnQuit.Click
        Me.Close()
    End Sub
End Class
```

Running Surface Stiffness and Tire Ellipticity Requirements for Adequate Ride Comfort

1. Stiffness of the Running Surface

The running surfaces for the system under consideration[20] is a pair of angles each of which is 8 inches wide, 6 inches high, and a 1/2th inch thick. From the *Manual of Steel Construction*, Third Edition, page 1-35, the moment of inertia in the vertical direction is I = 21.7 in^4. From page 1-98 the torsional constant is J = 0.584 in^4.

2. Natural Frequency of the Running Surface

From *Marks' Standard Handbook for Mechanical Engineers,* 10th Edition, page 3-73, Table 3.4.6 the fundamental radial frequency of a simply supported beam is

$$\omega_1 = \pi^2 \sqrt{\frac{EI}{ml^4}} \quad \text{where} \quad m = \frac{wA}{g} \qquad (1)$$

in which E is the modulus of elasticity [29.5(10)^6 psi for steel], I is the minimum moment of inertia of the angle in its weakest direction [$Ar^2 = 6.80(1.30)^2 = 11.49\ in^4$], l is the distance between supports (54 inches for the system under consideration), w is the weight density (0.283 lb/in^3 for steel), A is the cross sectional area (6.80 in^2 for the running surface of section 1), and g is the acceleration of gravity. The natural frequency $f = \omega / 2\pi$ cycles per second. Thus the fundamental natural frequency of our simply supported beam is

$$f_1 = \frac{\pi}{2l^2}\sqrt{\frac{EIg}{wA}} = \frac{\pi}{2(54)^2}\sqrt{\frac{29.5(10)^6(11.49)(32.17 \times 12)}{(0.283)(6.80)}} = 140.5Hz \qquad (2)$$

The excitation frequency, called the crossing frequency, is

$$f = \frac{V}{l}$$

in which V is the speed of the vehicle. Thus the critical speed for the running surface is

$$V = fl = (140.5/sec)(54/12ft)(60/88) = 431\text{mph}$$

[20] J. E. Anderson, "The Structural Properties of a PRT Guideway."

which is a great deal higher than any speed of interest. Thus, at urban speeds we can assume that the static deflection of the beam is also the deflection under our moving load.

3. Deflection of the Running Surface

From equation (28) of my paper "Deflection of a Continuous Beam resting on Regularly-Spaced Simple Supports under a Concentrated Load" the deflection under the front tires of a vehicle with the front tires halfway between supports and the rear tires on a support is

$$y_f = 0.2573\frac{P_f l^3}{48EI} = \frac{(0.2573)(54)^3}{48(29.5)(10)^6(21.7)}P_f = \frac{1.319}{(10)^6}P_f\ in$$

The wheel base WB is 1.5*l* or 81", which means that when the front wheels are midspan, the back wheels are over a support. The gross weight of our vehicle will be about 2000 lb of which half will be on the front wheels. Thus, let $P_f = 1000$ lb. Then $y_f = 0.0013$ in.

4. Vertical Acceleration due to Running Surface Deflection

Approximate the running surface between a pair of supports by a sine curve. Then

$$y = \frac{y_f}{2}\sin 2\pi\frac{Vt}{l}$$

in which V is the speed of travel and $t = 0$ as the vehicle passes over the left hand support if the motion is to the right. So $y = 0$ when $t = 0$ and $y = 0$ again when $Vt = l/2$. If we differentiate the deflection twice with respect to time t we get the maximum vertical acceleration, which is

$$|\ddot{y}_{max}| = \frac{y_f}{2}\left(\frac{2\pi V}{l}\right)^2 = 2\pi^2 f_c^2 y_f = 2\pi^2 f_c^2(0.0013\ in) = 0.0257 f_c^2\ \text{in/sec}^2 \quad (6)$$

in which $f_v = V/l$ is the crossing frequency. Assume $V = 60$ mph $= 88$ ft/sec, $f_c = 88(12)/54 = 19.56$ Hz. Then

$$|\ddot{y}_{max}| = \frac{0.0257}{12}(19.56)^2 = 0.819\frac{ft}{sec^2} = 0.025\ g$$

This is the peak acceleration. The r.m.s acceleration is less by the square root of two. Thus the r.m.s. acceleration is 0.02 g.

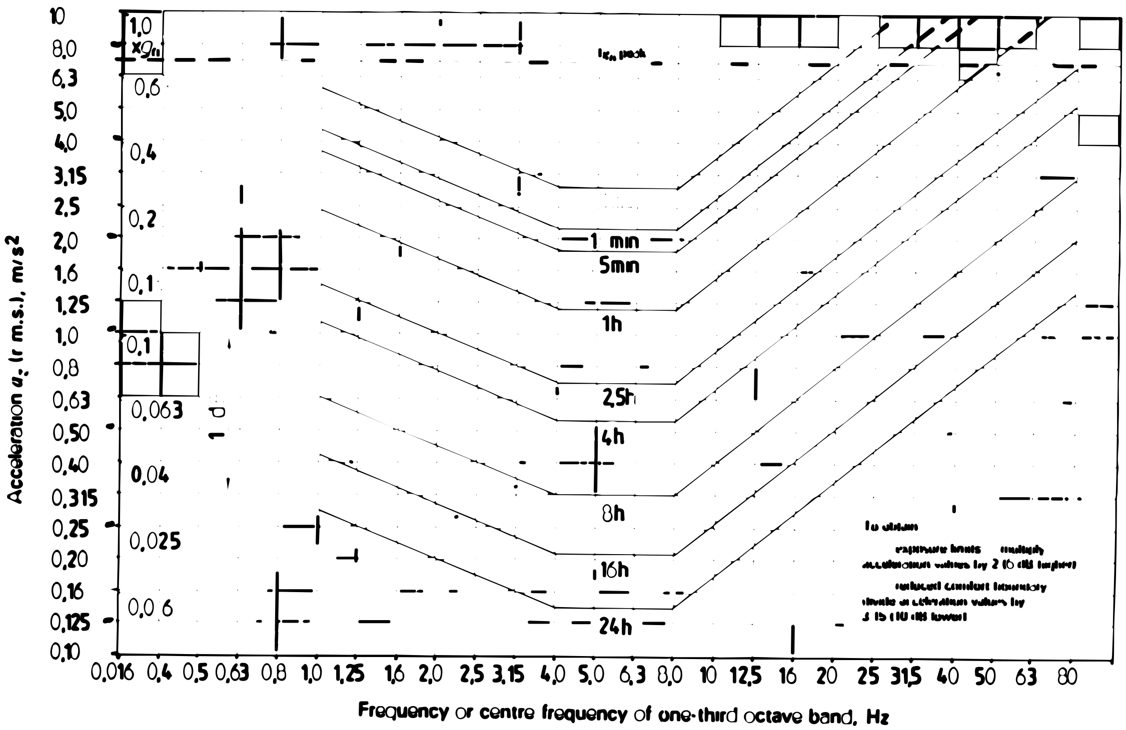


FIGURE 3-1 SPECTRAL COMPOSITION OF ACCELERATION

The above chart is from the ISO ride comfort standards. It can be seen that a point at 20 Hz and an r.m.s. acceleration of 0.02g is well under the curve for 24 hours of oscillation. At 20 Hz, the 1 hr duration curve is almost ten times higher. At 30 mph, the crossing frequency is 9.8 Hz and the r.m.s acceleration is about 0.006g, which is below the minimum values on the above chart. Our conclusion is that with the design we have selected, ride comfort will not be compromised due to deflection of the running surface.

5. Tire Ellipticity

In the book *Mechanics of Pneumatic Tires*, U. S. Department of Transportation, U. S. Government Printing Office, 1981, there are two chapters that discuss tire non-uniformity in terms of radial force variations. On page 616 one finds the statement: “Radial force variations acceptable for car manufacturers range from 100 to 150 N for the peak-to-peak value for European tire sizes from 13” or 14” rim diameters.” The mode shapes of variations in the deflection of a specific tire are shown in the following Figure 9.4.24 from *Mechanics of Pneumatic Tires.* The first mode is for the case in which the tire deflection varies from a maximum to a minimum and back to a maximum in one half a circumference. To obtain an understanding of tolerable variations, let us assume that the tire radius varies in the same

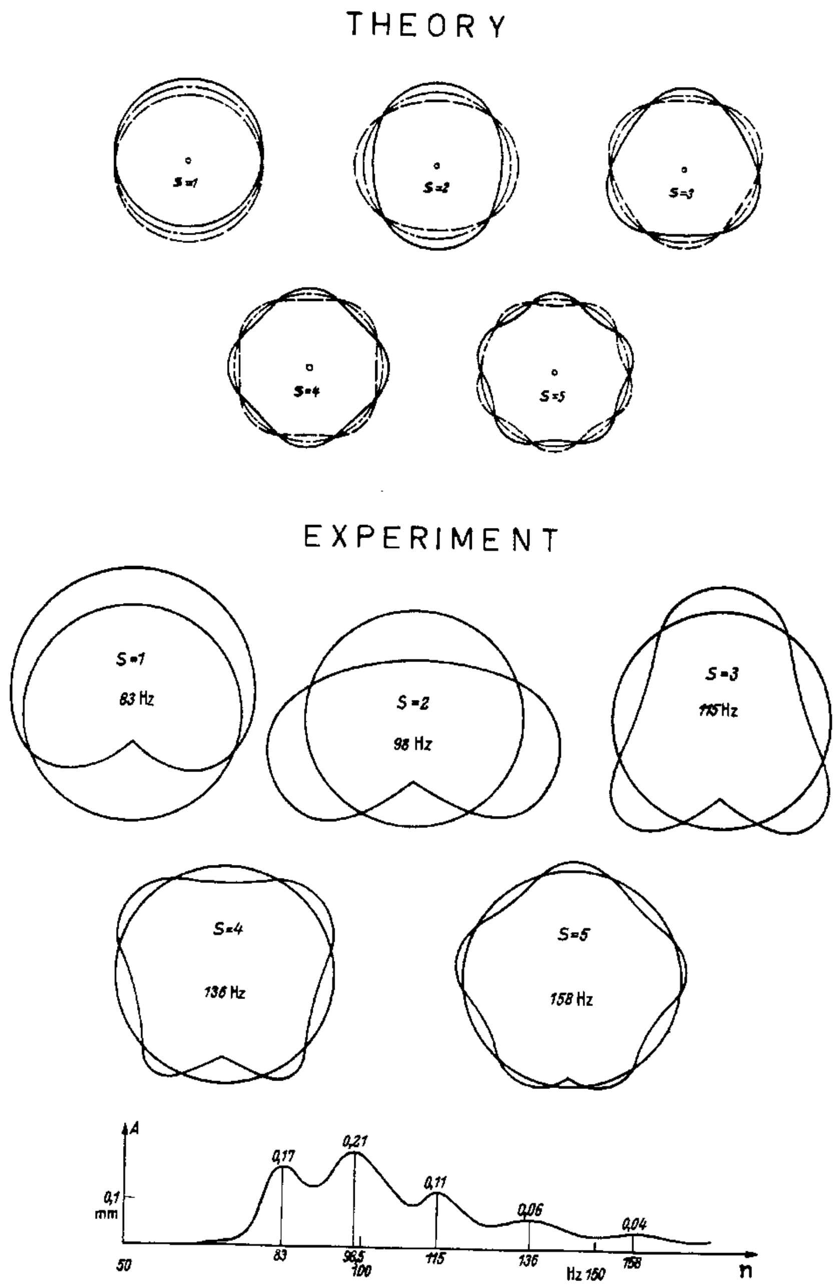


FIGURE 9.4.24. *Theoretical and experimental mode shapes and measured frequency response characteristic of the radial tread band deflections with respect to a radial excitation force for a radial-ply Michelin X 135-13 tire with p_i = 1.25 bar (from Böhm [33]).*

way. Let D be the nominal outer diameter of a tire so that πD is its circumference. Then let us assume that the tire radius R varies by an amount ΔR from a nominal value R_0 according to the equation

$$R = R_0 + \Delta R\left(\cos 2\pi \frac{x}{\pi R_0} + rnd\right) \tag{7}$$

in which $x = Vt$ is the distance traveled, V is the speed, t is the running time variable, and rnd is a random number from 0 to 1, introduced to take into account that peaks and valleys in the radius of the four tires are not coordinated.

The frequency of the variation is

$$f = \frac{V}{\pi R_0}. \tag{8}$$

6. Vertical Acceleration due to Tire Ellipticity

Assume for a moment for simplicity that our vehicle is a unicycle. Then, if it is at rest at time zero the vertical force balance between the vehicle's weight W and the tire force is $W = k\delta_o$, where δ_o is the static deflection of the tire at $x = t = 0$ and k is the tire stiffness. Let z_o be the height of the center of mass of the vehicle above a reference plane at $x = t = 0$ when the deflection at this point is zero. Then, the height of the center of mass of the vehicle above the reference plane at $x = t = 0$ is $z(0) = z_o - \delta_o$. If the vehicle moves to the right at speed V Newton's second law of motion gives

$$\frac{W}{g}\ddot{z} = -W + k\left(z_o - z\right)$$

or

$$\ddot{z} = -g + \frac{kg}{W}\left(z_o - z\right) = g\left[-1 + \frac{1}{\delta_o}\left(z_o - z\right)\right] \tag{9}$$

The following program solves equation (9). Since we plan to use 13" OD tires, let us consider the case when the radial force varies up to plus or minus approximately 100 N from the nominal value of 350lb(4.448N/lb) = 1557 N. This amount of force variation is achieved in the program if we let the quantity MaxVar = 0.0023Radius0. With this value, the program calculates the following results.

V	f	dTFmax	dTFmin	Amax	Amin	1 hr Comfort
mph	Hz	N		g r.m.s.	g r.m.s.	g r.m.s.
20	16.91	109.34	-112.08	0.050	-0.051	0.2
30	25.37	99.93	-102.62	0.045	-0.047	0.35
40	33.82	97.14	-99.82	0.044	-0.045	0.5

The last column gives, from the chart on page 3, the comfort value of vertical acceleration for a one-hour exposure at the indicated frequency. We see that the calculated values are a small fraction of these comfort limits, which means that tire Ellipticity is not a factor in ride comfort in even this extreme case in which the variable tire forces would act directly on the passenger as if the passenger rode directly over the tires and there is no other cushioning. Actually, the seat will provide some cushioning so the above calculations give higher vertical accelerations than would be experienced by the passengers. The other reason for concern would be the effect of the calculated vibratory acceleration on equipment that will be mounted either in the chassis or the cabin of the vehicle. A specification then is that the equipment be able to operate indefinitely in the above calculated vibratory environment.

```
'This program VERTACCL.BAS calculates the vertical acceleration
'of a vehicle due to tire irregularities
'Units are lb, in, sec

DEFDBL A-Z
DIM Counter AS INTEGER
DIM n AS INTEGER
Pi = 4 * ATN(1)
g = 9.80665 / .3048 * 12        'acceleration of gravity
NperLB = 4.448                  'Newtons per lb
W = 350                         'sprung mass on one tire
m = W / g                       'mass of vehicle
D = 13.25                       'diameter of tire
Circ = Pi * D                   'circumference of tire
Radius0 = D / 2                 'nominal tire radius
V = 30 * 88 / 60 * 12           'forward speed of vehicle
Del0 = .25                      'nominal tire deflection under load
k = W / Del0                    'spring constant of tire

dt = .001                       'computational time increment
dX = V * dt                     'computational distance increment

CLS
SCREEN 9
COLOR 7, 8
T0 = 20
Y0 = 200
scaleT = 1000
scaleA = 500
```

```
scaleV = 500
scaleZ = 60000
LINE (T0, Y0)-(T0, 0)
LINE (T0, Y0)-(640, Y0)

RANDOMIZE TIMER
R = RND
R = 0
t = 0
XX = 0
z.. = 0
z. = 0
MaxVar = .0023 * Radius0        'maximum variation in the tire radius
n = 5
Xmax = n * Circ
Xstart = Radius0
Radius = Radius0
z = Radius - Del0
Counter = 20
dTFmax = 0
dTFmin = W
Amax = 0
Amin = 0

DO
  X = V * t

  'Increase the tire variation gradually to settle initial conditions
  IF X < Xstart THEN
     Variation = 0
  ELSEIF X >= Xstart AND X < Xmax + Xstart THEN
     Variation = .5 * MaxVar * (1 - COS(Pi * ((X - Xstart) / Xmax)))
  ELSE
     Variation = MaxVar
  END IF

  'Define the variation in the tire radius
  Radius = Radius0 + Variation * COS(2 * Pi * (XX / Pi / Radius0 + R))
  Deflection = Radius - z
  TireForce = k * Deflection

  'Capture the maximum and minimum tire forces
  IF TireForce - W < dTFmin THEN
     dTFmin = TireForce - W
  ELSEIF TireForce - W > dTFmax THEN
     dTFmax = TireForce - W
  END IF

  'Solve the differential equations of motion
  z..old = z..
  z.. = -g + TireForce / m
  z.old = z.
  z. = z. + .5 * dt * (3 * z.. - z..old)
  z = z + .5 * dt * (z. + z.old)

  'Capture the maximum and minimum vertical accelerations
  IF z.. / g > Amax THEN
```

```
      Amax = z.. / g
   ELSEIF z.. / g < Amin THEN
      Amin = z.. / g
   END IF

   'To step through the program
   IF Counter = 20 THEN
      Counter = 0
      'SLEEP
   END IF
   Counter = Counter + 1

   PSET (T0 + scaleT * t, Y0 - scaleA * z.. / g), 10
   PSET (T0 + scaleT * t, Y0 - scaleV * z.), 11
   PSET (T0 + scaleT * t, Y0 - scaleZ * (Radius0 - Del0 - z)), 12

   XX = XX + dX
   IF XX >= Pi * Radius0 THEN XX = 0
   t = t + dt
LOOP UNTIL t > 5

PRINT "    Frequency, Hz: ";
PRINT USING "#######.##"; V / Pi / Radius0
PRINT "    dTFmax, dTFmin, Newtons: ";
PRINT USING "#######.##"; dTFmax * NperLB; dTFmin * NperLB
PRINT "    Amax, Amin, gs rms: ";
PRINT USING "#######.###"; Amax / SQR(2); Amin / SQR(2)
```

The Bending Moment at the Guideway Joint

Abstract

Every guideway section in ITNS must have an expansion joint, and it would be desirable if this joint need carry only shear. In a clamped beam under a uniform load, the bending moment is zero at the 21% point and the 79% point in the beam. The purpose of this paper is to calculate how close to a zero bending moment it is practical to design the expansion joint. In the ITNS guideway design, it is practical to place the joint at the 20% point, and we need to know what moment that joint need carry. The results developed in this paper show that the joint need carry only 6% of the maximum moment in a fully loaded guideway.

Consider a beam of length L clamped at both ends under a uniform load w lb/ft and a concentrated load P a distance a from the left end. Represent the beam in an orthogonal x-y reference frame with x along the length of the beam. $x = 0$ at the left end and $y = 0$ at the left support positive downward. The bending moment in the beam is

$$EI\frac{d^2y}{dx^2} = M(x) = M_1 - R_1 x + \frac{wx^2}{2} + P\langle x-a\rangle$$

in which M_1 is the moment at the left end and R_1 is the reaction force on the left support.

$$\langle x-a\rangle = x-a \text{ if } x > a \text{ and } 0 \text{ if } x \le a.$$

Integrating once and taking into account that the slope is zero at $x = 0$ we have

$$EI\frac{dy}{dx} = M_1 x - R_1\frac{x^2}{2} + \frac{wx^3}{6} + \frac{P\langle x-a\rangle^2}{2}$$

Since the slope is also zero at $x = 0$, we have

$$M_1 L - R_1\frac{L^2}{2} + \frac{wL^3}{6} + \frac{P(L-a)^2}{2} = 0, \qquad M_1 = R_1\frac{L}{2} - \frac{wL^2}{6} - \frac{P(L-a)^2}{2L}$$

Integrating once more

$$EIy = M_1\frac{x^2}{2} - R_1\frac{x^3}{6} + \frac{wx^4}{24} + \frac{P\langle x-a\rangle^3}{6}$$

Since $y = 0$ at $x = L$ we have

$$M_1 = R_1\frac{L}{3} - \frac{wL^2}{12} - \frac{P(L-a)^3}{3L^2} = R_1\frac{L}{2} - \frac{wL^2}{6} - \frac{P(L-a)^2}{2L}$$

Thus

$$\frac{R_1 L}{6} = \frac{wL^2}{12} + \frac{P(L-a)^2}{6L}\left[3 - 2\left(\frac{L-a}{L}\right)\right]$$

Therefore

$$R_1 = \frac{wL}{2} + \frac{P(L-a)^2(L+2a)}{L^3}$$

and

$$M_1 = \left[\frac{wL}{2} + \frac{P(L-a)^2(L+2a)}{L^3}\right]\frac{L}{2} - \frac{wL^2}{6} - \frac{P(L-a)^2}{2L}$$
$$= \frac{wL^2}{12} - \frac{P(L-a)^2}{2L}\left[1 - \frac{(L+2a)}{L}\right] = \frac{wL^2}{12} + \frac{P(L-a)^2 a}{L^2}$$

Hence

$$EIy(x) = \left\{\frac{wL^2}{12} + \frac{P(L-a)^2 a}{L^2}\right\}\frac{x^2}{2} - \left[\frac{wL}{2} + \frac{P(L-a)^2(L+2a)}{L^3}\right]\frac{x^3}{6} + \frac{wx^4}{24} + \frac{P\langle x-a\rangle^3}{6}$$
$$= \frac{wx^2}{24}(L^2 - 2Lx + x^2) + \frac{Px^2(L-a)^2}{6L^3}[3aL - Lx - 2ax] + \frac{P\langle x-a\rangle^3}{6}$$
$$= \frac{w(L-x)^2 x^2}{24} + \frac{Px^2(L-a)^2}{6L^3}[3aL - Lx - 2ax] + \frac{P\langle x-a\rangle^3}{6}$$

$$EIy\left(\frac{L}{2}\right) = \frac{wL^4}{384} + \frac{P(L-a)^2}{48}(4a - L)$$

If $a = \frac{L}{2}$

$$EIy\left(\frac{L}{2}\right) = \frac{wL^4}{384} + \frac{PL^2}{48 \times 4}(4a - L) = \frac{wL^4}{384} + \frac{PL^3}{192}$$

which agrees with results presented in well-known textbooks on the analysis of structures. Thus, the moment is given by

$$M(x) = \frac{wL^2}{12} + \frac{P(L-a)^2 a}{L^2} - \left[\frac{wL}{2} + \frac{P(L-a)^2(L+2a)}{L^3}\right]x + \frac{wx^2}{2} + P\langle x-a\rangle$$
$$= \frac{w}{12}(L^2 - 6Lx + 6x^2) + \frac{P(L-a)^2}{L^3}(La - Lx - 2ax) + P\langle x-a\rangle$$

If $P = 0$ then the moment vanishes when

$$\left(\frac{x}{L}\right)^2 - \frac{x}{L} + \frac{1}{6} = 0$$

The solution of which is

$$\frac{x}{L} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{2}{3}}\right) = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{3}}\right) = 0.7887, 0.2113$$

Place the moment equation in dimensionless form, and in so doing let $\alpha = \frac{x}{L}$ *and* $\beta = \frac{a}{L}$. Then, the moment equation becomes

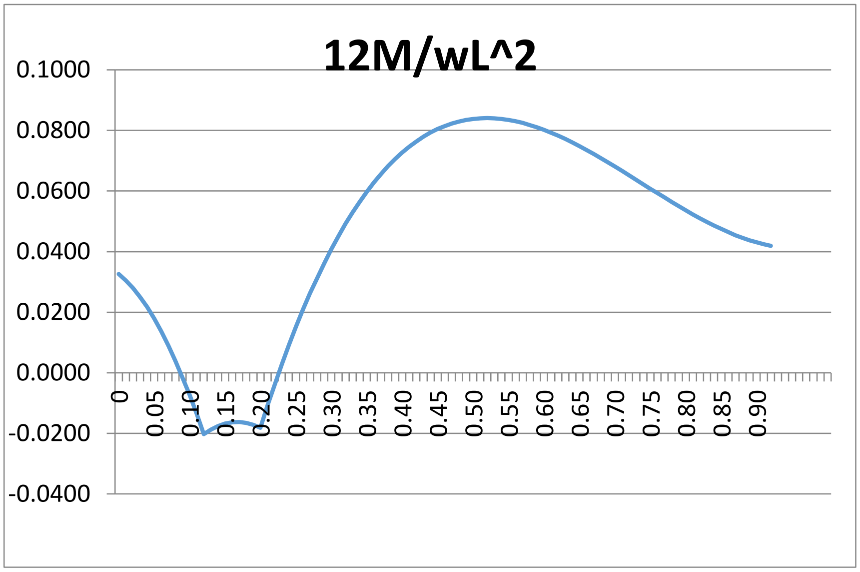
$$
\begin{aligned}
M(\alpha,\beta) &= \frac{w}{12}(L^2 - 6Lx + 6x^2) + \frac{P(L-a)^2}{L^3}(La - Lx - 2ax) + P\langle x - a\rangle \\
&= \frac{wL^2}{12}(1 - 6\alpha + 6\alpha^2) + PL(1-\beta)^2(\beta - \alpha - 2\alpha\beta) + PL\langle \alpha - \beta\rangle \\
&= \frac{wL^2}{12}\left\{1 - 6\alpha + 6\alpha^2 + \frac{12P}{wL}[(1-\beta)^2(\beta - \alpha - 2\alpha\beta) + \langle \alpha - \beta\rangle]\right\}
\end{aligned}
$$

Where $\frac{p}{wl}$ is the weight of a vehicle divided by the weight of the beam. If there are several point loads, then result is obtained by superposition. Let β_i be the *i*-th of a series of equal point loads. In this case

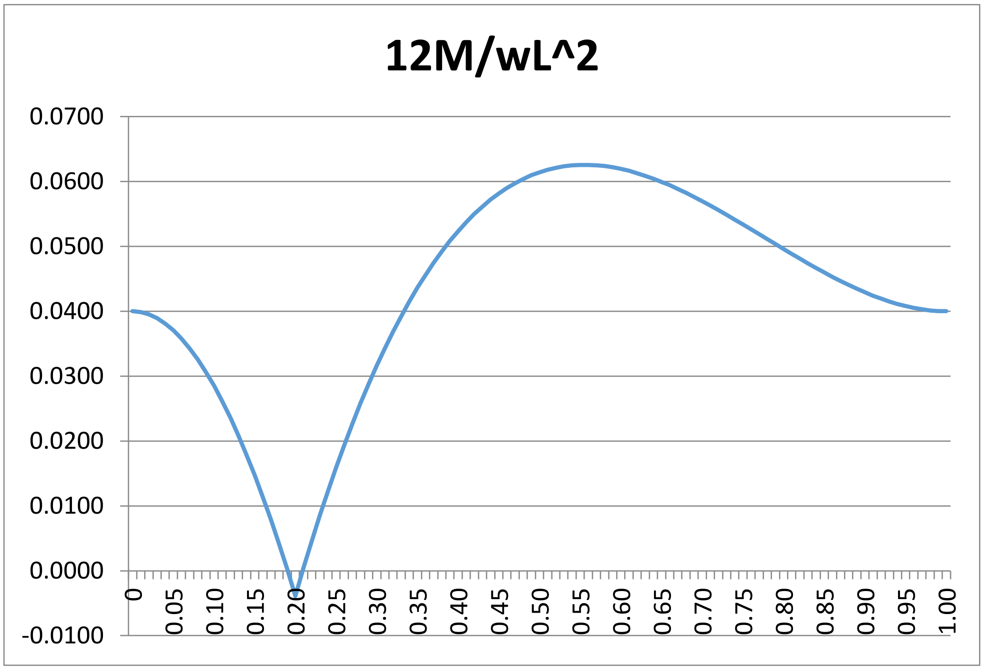
$$
\frac{12M}{wL^2} = 1 - 6\alpha + 6\alpha^2 + \frac{12P}{wL}\sum_{i=1}^{n}[(1-\beta_i)^2(\beta_i - \alpha - 2\alpha\beta_i) + \langle \alpha - \beta_i\rangle]
$$

Note that $\frac{wL^2}{12}$ is the maximum bending moment in a beam under uniform load *w*, and it occurs at $x = 0$ and $x = L$. Let the wheel base of the ITNS vehicle be *WB*. Then we can take the point loads in pairs separated by *WB*, with $\frac{WB}{L} = \frac{84}{90\times12} = 0.0778 \approx 0.08$ for the ITNS vehicle. The ratio $\frac{12P}{wL} = \frac{6W_{vehicle}}{wL} = \frac{6(1800)}{140\times90} = \frac{6}{7} = 0.8571$. It will be convenient to place the expansion joints at $\alpha = 0.2$ or 0.8. In the accompanying Excel Spreadsheet we have calculated the ratio $\frac{12M}{wL^2}$ for all relevant values of β_i. The results use the following input parameters:

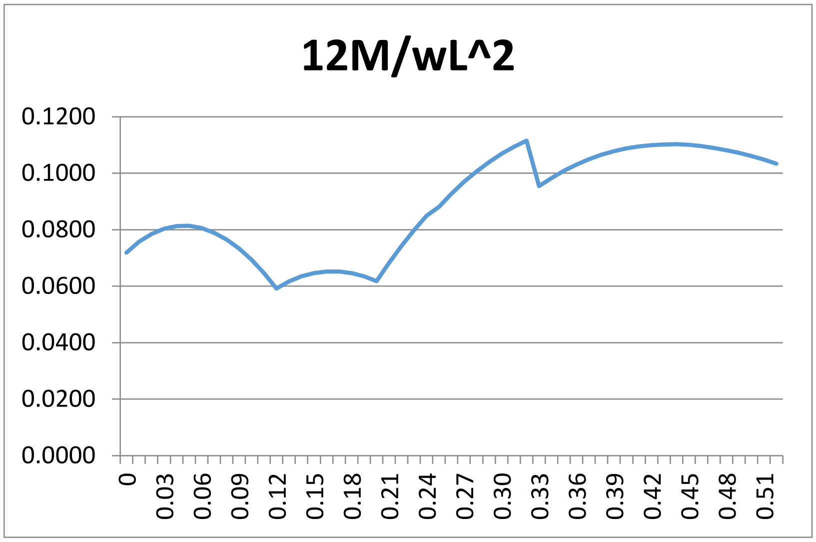
The Moment in a Beam at the Expansion Joints		
Beam Unit weight:	140	lb/ft
Vehicle Gross Weight:	1800	lb/ft
Beam Length:	90	ft
12P/wL:	0.8571	
Alpha:	0.2	
1-6A+6A^2:	0.0400	
WheelBase:	7.0000	
WB/L:	0.0778	
Minimum Headway:	0.5	sec
Speed:	30	mph
Vehicle Spacing:	22.0	ft
Spacing/L:	0.24	



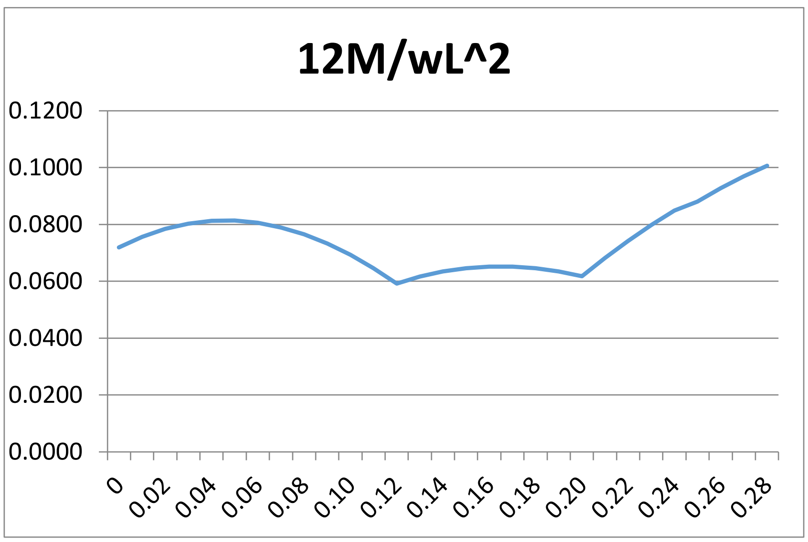
This first result assumes a one-point load due to half a vehicle.



This second result is for one vehicle with a pair of point loads one Wheel Base apart. Note that the maximum moment increased by about 0.02 to about 0.085.



This third result is for a pair of vehicles spaced 22 feet apart. We see that the maximum moment increased to about 11%. The curve is calculated only to $\beta = 0.76$ because to go further the forward vehicle would have passed the guideway support post.



This fourth result is for three vehicles each spaced 22 feet apart for a total spread of 44 feet. We see that while the area under the moment curve increased, the maximum increased hardly at all. This fifth result is for four vehicles covering from nose-to-nose 66 feet. We see that the maximum moment decreased slightly.

<u>Conclusion</u>

In designing the expansion joint at the 20% point, we will need to assume that the maximum moment it need carry is no more than 12% of the maximum moment in an unloaded beam. However, we have taken for the design load not only the guideway weight, but the guideway weight plus a root-mean-square load of vehicles on the guideway, i.e., 140 lb/ft + 1800/9/square root of

2, i.e. a total load of double the guideway weight per foot. Hence the expansion joint need carry only 6% of that maximum bending moment that the guideway must carry.

The Joint between Guideway Sections

Summary

The first thing a potential purchaser of *ITNS* will notice in his or her first ride on our test system will be the ride comfort. One of the most important considerations in ride comfort will be the manner in which the joints between guideway sections are aligned. Thus, the manner of treatment of these joints deserves very careful analysis. If, before a potential user is given an opportunity to ride, we judge that the ride comfort is inadequate, it is possible that we would have to tear down the entire guideway and replace it, for which there will likely be no money. The subject of this paper is an introduction to the consideration of these joints. How acceptable the ride comfort will be with our joint design can only be determined from following the movement of vehicles in our test guideway, but if we have not adequately accounted for potential bumps at the joints, we may not get a second chance. This is one more case in which adequate engineering analysis prior to operation is essential.

The Expansion Joint

Steel expands at about $6.5(10)^{-6}$ in/in/°F. In designing *ITNS*, we need to consider a temperature range from about -40°F to +120 °F. (I am aware that in Mid-Eastern countries we will have to design for a maximum temperature of at least 150 °F but then the minimum temperature will also be higher.) Over a temperature difference of 160 °F, steel will expand about $6.5(10)^{-6}(160) = 1.04(10)^{-3}$ in/in. Thus a 90-ft guideway section will expand about $1.04(10)^{-3}(90)(12) = 1.12$ in.

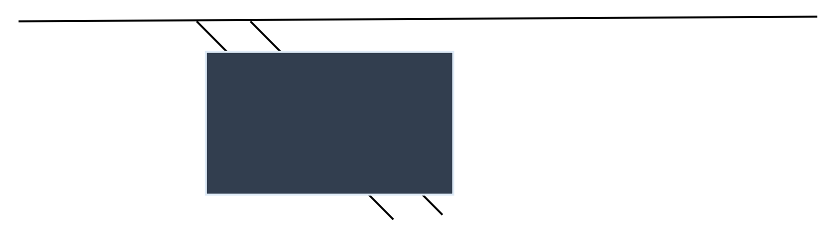


Figure 1. Running surfaces cut at a 45° angle, showing tire contact area.

It is possible for the expansion joint to be fingered, but fabricators have told me that a less expensive and likely satisfactory way is simply to cut the running surfaces at the joint at an angle, such as 45 deg. A 45 deg joint is illustrated in Figure 1, where the blue rectangle represents the contact area of the tire. The angle could be larger, up to perhaps 60 deg, but the advantage of 45 deg is that the complement of the angle is the angle, which removes one possible source of error. Also other fabrication problems may arise with a larger angle. We note that at any position of the tire-contact area, in passing over the expansion joint a portion of the tire contact area at each position along the direction of motion is in contact with one or the other of the two adjacent running surfaces, which minimizes the magnitude of a sudden vertical motion as the tire passes over the joint.

If the tire diameter is *D*, the width of the tire contact area *W*, and the tire deflection *d*, the contact area is obtained by application of the Pythagorean Theorem. Thus

$$Contact\ area = 2W\sqrt{d(D-d)}$$

Assume that in *ITNS* *W* = 3.5 in (the tire width is 4 in), *D* = 13.25 in, and *d* = 0.15 in. Then the contact area is 3.5 × 2.8 = 9.81 in^2. Assuming that the separation between adjacent running surfaces is zero at an outside temperature of 120^{o}F, we need to know the fraction of the contact area that touches either of the running surfaces at lower temperatures. This calculation is a straight forward application of geometry and is carried out by the program shown in Appendix A, where as an example, the calculation is carried out for outside temperatures of -40^{o}F, 0 ^{o}F, 40 ^{o}F, and 80 ^{o}F. The results for a 45^o joint are given in Figure 2. The minimum contact area as a fraction of the total contact area for joint angles of 45^o and 60^o are shown in the following table:

Outside temperature:	**-40^o F**	**0^o F**	**40^o F**	**80^o F**
45^o joint	0.684	0.760	0.840	0.920
60^o joint	0.815	0.861	0.907	0.954
Difference in Minimum Contact Area	16.1%	11.7%	7.4%	3.6%

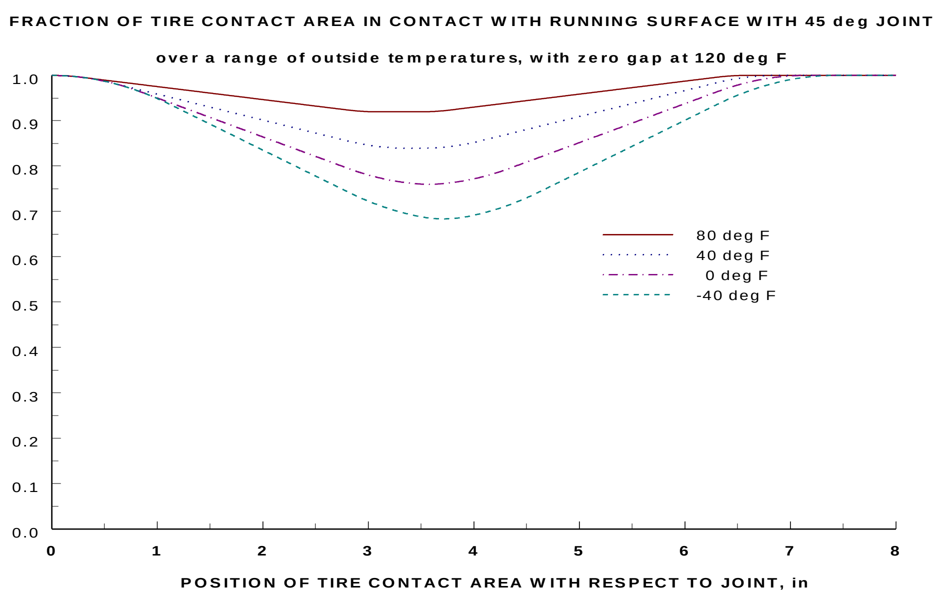


Figure 2. The fraction of the tire contact area that rests on one or the other of the two running surfaces.

<u>What are the consequences of passing over the gap in the running surfaces?</u> The distance the vehicle moves in completely passing over the gap is W + Contact Length + gap and with the above values, it would at -40^{o}F be 3.5 + 2.8 + 1.12 = 7.42 in. At a speed of 20 mph or 352 in/sec, the time required to pass the gap is about 20 millisec. A slight slapping sound may be

heard, the lower the outside temperature the louder. Also an extremely minor bump may be felt, but with even just seat suspension, it likely will not be noticed. We are accustomed to much larger bumps in automobile rides. Experimental data on the test track is needed. As a backup, we could build in some fingered joints so that we could compare.

<u>Consequences of a Step in the Running Surface</u>

It cannot be expected that the adjacent running surfaces of two 90-ft guideway sections will be aligned exactly. How much misalignment will be tolerable can best be determined by experiment. I have solved the equations of motion in passing over gap with a slope discontinuity[21], and calculated the maximum vertical acceleration that would be experienced by a passenger seating in a sprung seat. Unfortunately, there is no standard against which to state what maximum misalignment would be acceptable. The important point is that we will need to develop a way to keep the step change as close to zero as economically practical. This concern applies most importantly to the major running surfaces and to a lesser degree to the side wheels because the side loads will be smaller. Sudden upward or downward motion of the wheel does not translate directly into motion that will be felt by the passenger sitting on a sprung seat, and the step changes that can be produced in *ITNS* will be much smaller than experienced by an automobile passing over expansion joints in a highway.

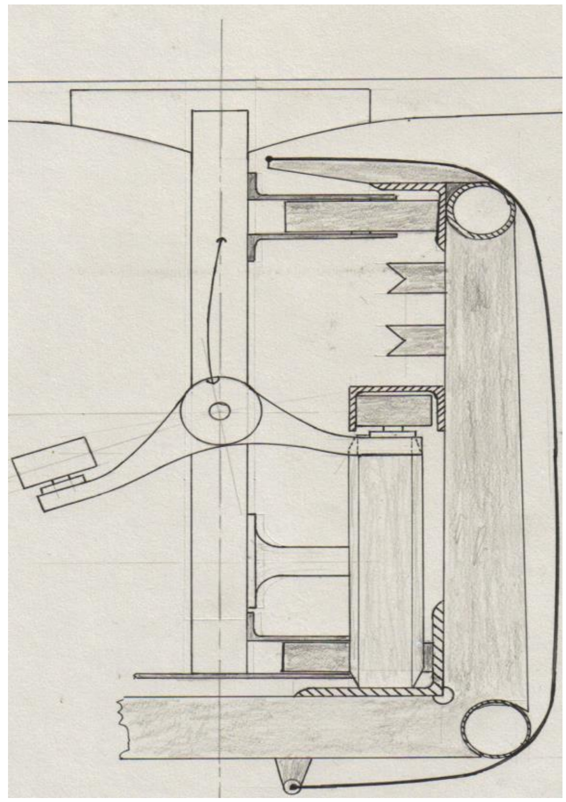


Figure 3. Guideway Cross Section.

<u>How can any step change in the running surface at the expansion joints be minimized?</u>

The cross section of the guideway, which is symmetrical about its vertical centerline, is shown in Figure 3. The main load is carried by four pipe stringers of 4 in OD each. The plan we have envisioned is that at the expansion joints, a pipe of slightly smaller OD than the ID of the each 4-inch pipe and from about ten to twelve feet long (the smallest acceptable length must be determined during the computer analysis of the guideway) will be slid half way inside the outer pipes with one end secured perhaps by a tack weld and the other allowed to slide. The sliding half should be covered by some kind of low-friction material. If a section of guideway would have to be replaced, these pipes must be cut at the joint. If nothing more is done, the maximum differ-

[21] See my paper "Ride Comfort over Slope Discontinuity."

ence in alignment of the adjacent running surfaces is dependent on a buildup of several tolerances, and these can be reduced by manufacturing key dimensions in the U-frames to as close a tolerance as would be commercially practical.

A steel fabricator asserted to me that it would be practical to hold the left-to-right dimension of the U-frames at the location of the upper and lower lateral wheels to within 0.005" to 0.010". I am convinced that that will be acceptable and is one reason for this particular design configuration, which will permit the upper and lower angle running surfaces to be pressure-welded to the U-frames − pressure-welded because of warping that occurs due to the heat generated by arc welding. Now, in addition, for the pair of U-frames that will be adjacent at a joint, the position of the cutouts for the longitudinal pipe-stringers with respect to the running surfaces must be closely controlled. If necessary U-frames to be placed in the end U-frames of adjacent sections could be marked and installed as matched pairs. Hopefully, no additional alignment procedure will be required.

The guideway sections are to be manufactured in 45-ft lengths, assembled in the field into the desired 90-ft sections. This will mean that the expansion joint will be at the 25% points between posts instead of the ideal 21%, but as shown in Appendix B, the bending moment at the 25% point is only one eighth of the maximum bending moment. The procedure for assembling two 45-ft lengths will be similar to assembling 90-ft sections to each other, except that here there is no expansion joint. If the running surfaces can't be aligned adequately based on the accuracy of the U-frames, the U-frames can be set back from the ends sufficiently so that the running surfaces can be brought into alignment by pressure welding sufficiently stiff angle sections to their back sides lapped between the adjacent running surfaces.

<u>In a loop guideway, how will the last 90-ft section be secured?</u>

In all sections except the final one, the procedure of sliding in the inside pipes into one end, securing them by a tack weld, and then sliding the next section on to these inside pipes is straightforward. What to do with the final section that closes the loop? There are two problems: 1) its length must be correct in order that the gap between adjacent sections not be too large or too small, and 2) the procedure just described for installing the inside pipes must be modified.

As to the length, would it be better a) to rely on sufficiently accurate surveying so that the remaining gap is within plus or minus say 0.20 inch of nominal, or b) to delay manufacture of the final section until the adjacent sections have been installed? For many reasons, we will have to plan that the sections cannot all be exactly 90 ft long, hence procedure b) may be the best.

<u>What should be the procedure for installing the inside pipes into the closing section?</u>

By not yet firmly bolting the post brackets (that connect the post to the guideway) that are close to one end of the final guideway section it will be possible to move this guideway section out of alignment sufficiently so that the procedure described for any other section will work for one of its ends. But then if the last section is drawn into alignment at the final joint, there is no way to insert the inside pipes. One solution would be to 1) cut the main pipe stringers short on the final section by say two feet and set back the end U-frame by the same amount. Then 2) before the final section is installed slide the inside pipes into each of the four stringers far enough so that their ends interface with the end of the section. 3) Bring the final joint into alignment and then 4) pull the inside pipes out half way so that an equal length remains in each of the two sections. Once done, one end of each of these inside pipes can be tack welded to maintain its position. This procedure appears satisfactory because the maximum bending moment at the expansion joint will, per the analysis in Appendix B, be no more than about one eighth of the maximum bending moment in the guideway, and thus it can be carried by the inside pipe stringers. This procedure works best if the final guideway section is one of the straight sections.

Appendix A

Program to calculate the tire contact area.

```
'This program GAP.BAS calculates the fraction of covered area in passing
'over a gap in the running surface.
DEFDBL A-Z
DEFINT I-J

Jmax = 1000
DIM FrnAreaCovered(1 TO 4, 0 TO Jmax)

TireWidth = 3.5    'tire width,inches
TireDiam = 13.25   'tire diameter, inches
TireDefl = .15     'tire deflection, inches
'From the Pythagorean Theorem:
ContactLength = 2 * SQR(TireDefl * (TireDiam - TireDefl))
'Tire contact area:
TCA = TireWidth * ContactLength

CoefOfExpansion = .0000065#     'Expansion coefficient of steel,in/in /deg F
ExpOfGdwy = CoefOfExpansion * 90 * 12  'Expansion of guideway per degree F

Angle = 60     'angle gap boundry makes with direction transverse to motion
Angle = Angle * ATN(1) / 45  'convert to radians
TwoTanAng = 2 * TAN(Angle)
'Distance from start of gap on one side of tire to start on other side:
TWtanAng = TireWidth * TAN(Angle)
'Distance in direction of motion from start of gap to end of gap:
TWtnpCL = TWtanAng + ContactLength

CLS
SCREEN 9
COLOR 7, 8
X0 = 10
Y0 = 300
Xscale = 60
CAscale = 250
LINE (X0, Y0)-(600, Y0)
LINE (X0, Y0)-(X0, 0)
LINE (X0, Y0 - CAscale)-(600, Y0 - CAscale)

dx = .01

OPEN "GAP45.ASC" FOR OUTPUT AS #1
```

```
FOR I = 1 TO 4
  DelT = 40 * I    '120 deg F - outside temperature, deg F
  gap = ExpOfGdwy * DelT  'separation between guideway sections
  x = 0    'x is the coordinate in the direction of motion, 0 just as
           'the tire contact area first touches the first part of the gap
  AC.old = TCA   'to determine minimum area covered
  Flag% = 0
FOR J = 0 TO Jmax
  IF x < gap THEN
     AreaCovered = TCA - x ^ 2 / TwoTanAng
  ELSEIF x < TWtanAng THEN
     xmg = (x - gap) ^ 2 / TwoTanAng
     IF x > ContactLength THEN
        xml = (x - ContactLength) ^ 2 / TwoTanAng
     ELSE
        xml = 0
     END IF
     IF x > ContactLength + gap THEN
        xmlg = (x - ContactLength - gap) ^ 2 / TwoTanAng
     ELSE
        xmlg = 0
     END IF
     AreaCovered = TCA - x ^ 2 / TwoTanAng + xmg + xml - xmlg
  ELSEIF x < TWtanAng + gap THEN
     xmg = (x - gap) ^ 2 / TwoTanAng
     IF x > ContactLength + gap THEN
        xmCLmG = (x - ContactLength - gap) ^ 2 / TwoTanAng
     ELSE
        xmCLmG = 0
     END IF
     AreaCovered = (TWtnpCL - x) ^ 2 / TwoTanAng + xmg - xmCLmG
  ELSEIF x < TWtnpCL + gap THEN
     IF TWtnpCL > x THEN
        WCLmx = (TWtnpCL - x) ^ 2 / TwoTanAng
     ELSE
        WCLmx = 0
     END IF
     AreaCovered = TCA - (TWtnpCL + gap - x) ^ 2 / TwoTanAng + WCLmx
  ELSE
     AreaCovered = TCA
  END IF

  'To determine minimum covered area, AC.min:
  IF Flag% = 0 AND AreaCovered > AC.old THEN
     AC.min = AC.old / TCA
     Flag% = 1
  END IF
  AC.old = AreaCovered

  FrnAreaCovered(I, J) = AreaCovered / TCA
  PSET (X0 + Xscale * x, Y0 - CAscale * FrnAreaCovered(I, J))
  x = x + dx
NEXT J
PRINT " Minimum Covered Area: ";
PRINT USING "###.###"; AC.min
NEXT I
FOR J = 0 TO Jmax
```

```
 WRITE #1, J * dx, FrnAreaCovered(1, J), FrnAreaCovered(2, J), FrnAreaCov-
ered(3, J), FrnAreaCovered(4, J)
NEXT J
CLOSE #1
```

Appendix B

The distribution of bending moments in the guideway.

Consider a beam of length L clamped at both ends and subject to a uniform load of w lb/ft. Let the origin of coordinates be located at the left end with x along the beam and y in the direction of the load. The bending moment in the beam at any point is

$$EI\frac{d^2y}{dx^2} = M - Rx + \frac{wx^2}{2}$$

in which E is the modulus of elasticity, I is the moment of inertia, M is the as yet unknown bending moment at the end of the beam, and $R = wL/2$ is the reaction force at each end of the beam. Integrate once and take into account that with a clamped beam the slope is zero at each end. Then

$$EI\frac{dy}{dx} = Mx - R\frac{x^2}{2} + w\frac{x^3}{6}$$

At the right end of the beam where $x = L$, the slope is also zero. Thus, dividing by L we find that

$$M = w\frac{L}{2}\frac{L}{2} - w\frac{L^2}{6} = w\frac{L^2}{12}$$

Hence the moment distribution in the beam is

$$M(x) = \frac{wL^2}{2}\left(\frac{1}{6} - \frac{x}{L} + \frac{x^2}{L^2}\right)$$

By setting $M(x)$ to zero, we find the points in the beam where the bending moment is zero. Thus

$$\left(\frac{x}{L}\right)^2 - \frac{x}{L} + \frac{1}{6} = 0$$

Thus

$$\frac{x}{L} = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{3}}\right) = 0.211, 1 - 0.211$$

We also note that

$$M\left(\frac{L}{4}\right) = \frac{wL^2}{2}\left(\frac{1}{6} - \frac{1}{4} + \frac{1}{16}\right) = -\frac{wL^2}{96}, \qquad M\left(\frac{L}{2}\right) = -\frac{wL^2}{24}$$

We thus see that the moment at the quarter point is only 12/96 =1/8th of the maximum moment, which is at the ends of the beam. It will thus be easily carried by the four inside pipes at the quarter points.

A Dynamic Analysis of the Switch Rail Entry Flare

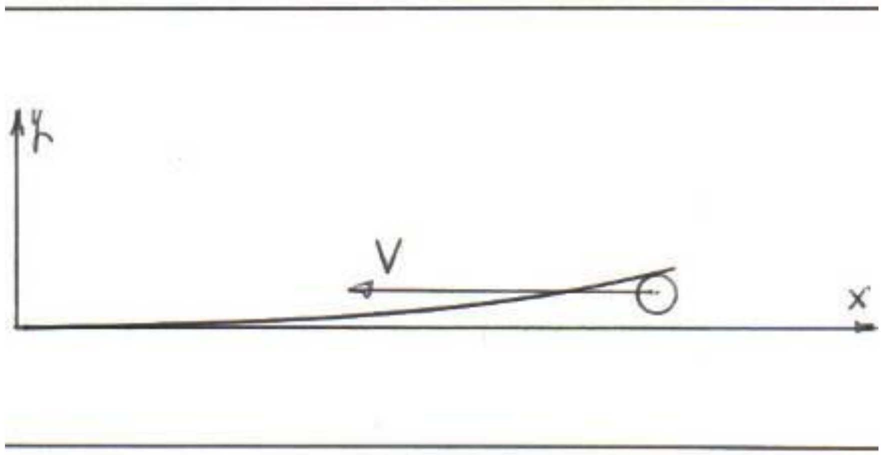


Figure 1. A switch wheel entering the flared switch rail.

The Problem

Figure 1 illustrates a switch wheel attached to an ITNS vehicle moving to the left at speed V and engaging a flared switch rail. When the wheel hits the switch rail it will transfer to the vehicle, which is running above the rail, a sudden motion and will impart to the rail an impulsive force. We must know the magnitude of the lateral acceleration and jerk imparted to the vehicle and the magnitude of the impulsive force so that we can design the rail to have sufficient strength when subjected to a sequence of impulsive loads.

Analysis

$$If\ x \le x_1\ let\ y = y_1\left(\frac{x}{x_1}\right)^n\ \ elseif\ x > x_1\ let\ y = y_1 + n\frac{y_1}{x_1}(x - x_1)$$

i.e. let the slopes match at $x = x_1$.

Let the flare length be L_{flare} and let the maximum transverse deviation of the flare be D_{max}. Assume the vehicle enters the flare at $x = L_{flare}$ at a speed V and time $t = 0$, and the flare ends at $x = 0$. Then the position of the vehicle is $x = L_{flare} - Vt$. Then, in the region $0 \le x \le x_1$

$$\frac{dy}{dt} = ny_1\frac{x^{n-1}}{x_1^n}(-V), \qquad \frac{d^2y}{dt^2} = n(n-1)y_1\frac{x^{n-2}}{x_1^n}V^2$$

Let us restrict the lateral acceleration at $x = x_1$ to a_{limit}. Then at $x = x_1$ we have the equation

$$a_{limit} = n(n-1)\frac{y_1V^2}{x_1^2}$$

or

$$y_1 = \frac{a_{limit}}{n(n-1)V^2}x_1^2$$

At $x = L_{flare}$ we have

$$D_{max} = y_1 + n\frac{y_1}{x_1}\left(L_{flare} - x_1\right) = \frac{a_{limit}}{n(n-1)V^2}x_1^2\left[1 + n\left(\frac{L_{flare}}{x_1} - 1\right)\right]$$
$$= \frac{a_{limit}}{n(n-1)V^2}\left[-(n-1)x_1^2 + nL_{flare}x_1\right]$$

or

$$u = x_1^2 - \frac{n}{n-1}L_{flare}x_1 + \frac{nD_{max}V^2}{a_{limit}} = 0$$

the solutions of which are

$$x_1 = \frac{1}{2}\left[\left(\frac{n}{n-1}\right)L_{flare} \pm \sqrt{\left(\frac{n}{n-1}\right)^2 L_{flare}^2 - 4\frac{nD_{max}V^2}{a_{limit}}}\right]$$

Note that

$$\frac{du}{dx_1} = 2x_1 - \frac{n}{n-1}L_{flare}$$

which is zero at

$$x_1 = \frac{n}{2(n-1)}L_{flare}$$

For n = 2 $\quad x_1 = L_{flare}\ and\ for\ n = 3 \quad x_1 = \frac{3}{4}L_{flare}$.

This we note that the equation $u(x_1)$ has negative slope at $x_1 = 0$ and that x_1 for the plus root is at least a large fraction of the length of the flare. Thus, we take the solution to be

$$x_1 = \frac{1}{2}\left(\frac{n}{n-1}\right)L_{flare}\left[1 - \sqrt{1 - 4\frac{(n-1)^2D_{max}V^2}{na_{limit}L_{flare}^2}}\right]$$

If we take a_{limit} = 0.25g =8.05 ft/sec^2 and V = 44 ft/sec, then for a real solution we must have

$$4\frac{(n-1)^2D_{max}V^2}{na_{limit}L_{flare}^2} < 1\ or\ D_{max} < \frac{n}{(n-1)^2}\frac{a_{limit}L_{flare}^2}{4V^2}$$

A reasonable flare length would be about 8 ft, in which case we would have

$$D_{max} < \frac{n}{(n-1)^2}0.0665$$

If n = 2 then D_{max} < 0.133 ft or 1.6 in, and if n = 3 D_{max} < 0.050 ft or 0.6 in.

Alternative Shapes

1. The Rail is basically linear.

Let the transition rail be defined by the equations

$$If\ 0 \le x \le x_1\ let\ y = y_1\left(\frac{x}{x_1}\right)^2\ elseif\ x > x_1\ let\ y = y_1 + 2\frac{y_1}{x_1}(x - x_1)$$

Assume that the switch wheel hits the switch rail when $x = L > x_1$ and $t = 0$. Then the position of the switch wheel, moving to the left at constant speed V , is given by

$$x_{swx} = L - Vt \tag{1}$$

Then, with the vehicle moving to the left,

$$y = y_1 + 2\frac{y_1}{x_1}(L - Vt - x_1), \qquad \dot{y} = -2\frac{y_1}{x_1}V, \qquad \ddot{y} = 0\ if\ x > x_1 \tag{2}$$

If

$$x \le x_1\ \ y = y_1\left(\frac{L-Vt}{x_1}\right)^2, \qquad \dot{y} = -2y_1\frac{(L-Vt)V}{x_1^2}, \qquad \ddot{y} = \frac{2y_1}{x_1^2}V^2, \qquad \dddot{y} = 0.$$

Thus, if we wish to limit the lateral acceleration to say $\ddot{y} = ag$ we must select x_1 at

$$x_1 = V\sqrt{\frac{2y_1}{ag}}$$

Let L$\gg x_1$, and let the transverse displacement of the flare at $x = L$ be $y(L) = y_L \gg y_1$. For example, let L = 6 ft = 72 in and y_L = 0.5 in at t = 0. Let x_1 = 6 in. Then

$$y_1 = \frac{0.5}{1 + \frac{2(72-6)}{6}} = \frac{1}{46} = 0.022\ in$$

Thus, almost always, the switch wheel hits the flared rail when $y > y_1$, but if it hits the flared rail when $y < y_1$ and V = 35 mph =51.3 ft/sec = 616 in/sec then $\dot{y} \le \frac{2(616)}{46(6)} = 4.5\frac{in}{sec}$.

2. The Rail is a parabola.

First assume that the shape of the flared switch rail is described by a parabola, i.e.

$$y = y_l \frac{x^2}{l^2}$$

(A-1)

in which $x = l$ is the variable position where the switch wheel strikes the switch rail, and $y(l) = y_l$ is the lateral displacement of the switch rail with respect to its lateral displacement at $x = l.$

Substituting equation (A-2) into equation (A-1) we find the lateral displacement of the switch wheel as a function of time:

$$y(t) = y_l \frac{(l - Vt)^2}{l^2} \ if \ t \leq \frac{l}{V} \ \ or \ 0 \ \ if \ t > \frac{l}{V}$$

(A-3)

Thus the transverse velocity, transverse acceleration, and transverse jerk experienced by the switch wheel, arm and hence vehicle are given successively for $t \leq l/V$ by

$$\begin{aligned} Speed: &\qquad \dot{y} = -2y_l \frac{(l - Vt)}{l^2} V \\ Acceleration: &\qquad \ddot{y} = 2y_l \frac{V^2}{l^2} \\ Jerk: &\qquad \dddot{y} = 0 \end{aligned}$$

(A-4)

For $t > l/V$ the lateral motion is zero. So the vehicle experiences a sudden negative step change in lateral speed, followed by a linearly decreasing lateral speed to zero at $t = l/V$. The lateral acceleration suddenly receives a positive value at $t = 0$ and remains constant until $t = l/V$ whereupon it suddenly vanishes, thus at both the beginning and ending of the spiral, the vehicle experiences an infinite spike of jerk in zero time – in the physical world a very large pulse of jerk for a very short period of time.

3. The Switch Rail is a Cubic.

On the other hand, if the shape of the flare conformed to a cubic curve, we would find that the motion of the switch wheel for $t \leq l/V$ would conform to the following equations:

$$\begin{aligned} Position: &\qquad y = y_l \frac{(l - Vt)^3}{l^3} \\ Speed: &\qquad \dot{y} = -3y_l \frac{(l - Vt)^2}{l^3} V \\ Acceleration: &\qquad \ddot{y} = 6y_l \frac{(l - Vt)}{l^3} V^2 \\ Jerk: &\qquad \dddot{y} = -6y_l \frac{V^3}{l^3} \end{aligned}$$

(A-5)

So, in this case the switch wheel, and hence the vehicle, receives a sudden step change in lateral speed, which for a fixed value of l is 1.5 times as much as in the parabolic case, and a pulse of acceleration 3 times as much as in the parabolic case, which decreases linearly to zero at $t = l/V$. The jerk goes from zero when $t < 0$ to a finite value constant throughout the transition, and then suddenly drops to zero again when $t > l/V$. Since acceleration suddenly takes a finite value at $t = 0$, there occurs there an infinite spike of jerk.

In both cases there is an infinite spike of jerk at $t = 0$. The question then is as follows: Is it better to live with one spike of jerk rather than two? In both cases we see that the lateral motion is largest when $t = 0$. If J_c is the comfort value of jerk, we see from the last of equations (A-5) that the cubic flare length should be

$$l = V\left(\frac{6y_l}{J_c}\right)^{1/3} \tag{6}$$

For example, if $V = 16\,\text{m/s}$, $y_l = 3\,\text{mm}$, and $J_c = \frac{1}{4}\,\text{g/s}$, then $l = 3.11\,\text{m}$ (10.2 ft). For the same values, with the cubic curve, we see that

$$\ddot{y}_{\max} = 6y_l\frac{V^2}{l^2} = 6(0.003)\left(\frac{16}{3.11}\right)^2 = 0.477\,\text{m/s}^2 = 0.049\,g, \tag{7}$$

which is well below the comfort level of 0.25g. Thus with $l = 3.11$ m we experience a pulse of jerk corresponding to a very small step in acceleration.

How uncomfortable is such a small step in acceleration? Consider the following thought experiment: Suppose you are in an elevator and the elevator suddenly starts to accelerate downwards. While it doesn't suddenly drop as if the cable were cut, i.e., at an acceleration rate of one g, the motion is uncomfortable – one gets a funny feeling in the pit of the stomach. Now suppose you are sitting in a wagon on a plane inclined at an angle of say 10 deg or 0.175 radian restrained by a rope. Suppose someone suddenly cuts the rope. You will feel a smaller funny feeling, and it is clear that the smaller the incline the smaller the uncomfortable feeling. An acceleration given by equation (7) corresponds to an incline of 0.049 radian or 2.8 degrees.

With a parabolic flare, the maximum step change in acceleration is one third as much with the same flare length, but it occurs twice. To design with the parabolic flare, assume the maximum lateral acceleration reaches the comfort value of $A_c = 0.25g$. Then, from the second of equations (4) we get

$$l = V\left(\frac{2y_l}{A_c}\right)^{1/2} \tag{8}$$

Using the values assumed following equation (6) we would get

$$l = V\left(\frac{2y_l}{A_c}\right)^{1/2} = 16\left[\frac{2(0.003)}{\frac{1}{4}g}\right]^{1/2} = 0.792\,\text{m.} \tag{9}$$

but with this small a flare length, the step changes in acceleration correspond to cutting the rope on an incline of $0.25(180/\pi) = 14.3\,\text{deg}$ may be uncomfortable. If we were to assume say $A_c = g/8$ the flare length would be 1.12 m. If the flare length were the value $l = 3.11$ m calculate from equation (6), we see that if the transition shape is parabolic, we find from the second of equations (4) that

$$\ddot{y}_{\max} = 2(0.003)\left(\frac{16}{3.11}\right)^{1/2} = 0.0136\,\text{m/s}^2 = 0.0014\,g \tag{10}$$

We see that to make further progress we need to determine how large y_l must be, considering the most extreme motion of the vehicle subject to a combination of a maximum side wind, a maximum unbalanced load, and a maximum centrifugal force, all acting in the same direction. This information requires an accurate dynamic simulation of the lateral motion of the vehicle[22], and will depend on the tire stiffnesses assumed.

[22] This study is available.

The Maximum Stress in the Switch Rail

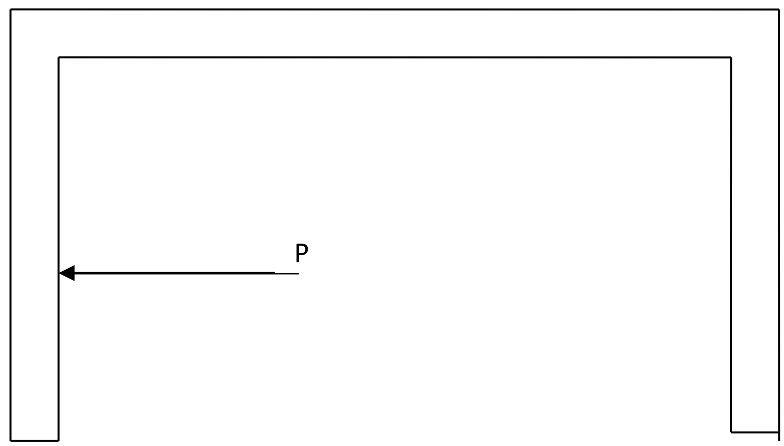


Figure 2. Cross section of the switch rail.

The inertia load of the switch wheel on the switch rail is

$$P = W\frac{\ddot{y}_{\max}}{g} \tag{11}$$

The load P produces a bending moment Pa where a is the distance from the point of application of the load to the centerline of the top horizontal portion of the switch rail, shown in Figure A-2. The maximum stress in the top member is the sum of the maximum bending stress and the tensile stress:

$$\begin{aligned} \sigma_{\max} &= \frac{Pa}{I/c} + \frac{P}{tw}, \quad \text{where} \quad \frac{I}{c} = \frac{wt^3}{12}\frac{2}{t} = \frac{wt^2}{6} \\ \therefore\ \sigma_{\max} &= \frac{P}{wt}\left(\frac{6a}{t} + 1\right) \end{aligned} \tag{12}$$

in which t is the thickness of the steel rail and *w* is the effective length into the paper in Figure A-2. At this point, without a finite-element analysis, we can only guess at *w*. Suppose $w = 6$ in and $t = 1/4$ in. Then $\sigma_{\max} = 96.7P$ psi if P is in pounds. Assume we select the flare length so that $\ddot{y}/g \le 1/8$. The weight we need consider is not the entire weight of the vehicle because there are switch wheels at the front and back of the vehicle. To be conservative assume W = 1500 lb. Then $p = 1500/8 = 187.5$ lb and $\sigma_{\max} = 18131$ psi. With mild steel the accepted maximum tensile stress is 15, 000 psi. Thus we must either use thicker sections or a higher grade of steel.

This is just a "ball park" analysis. To settle on the parameters, we need, as mentioned, both an accurate dynamic analysis of the motion of the vehicle passing through a merge or diverge section of guideway and a finite-element analysis on the switch rail.

The switch rails will be welded to the vertical U-frames of the guideway every 54 in. Thus we must consider three factors:

- The maximum bending stress in the rail.
- The maximum deflection of the rail
- The maximum stress at the points of attachment to the U-frames.

To obtain the first two of these quantities, we need the moment of inertia of the cross section of the rail shown in Figure A-2. It is

$$I = \frac{1}{12} tb^3 + 2dt\left(\frac{b}{2}\right)^2 = \frac{tb^3}{12}\left(1+6\frac{d}{b}\right) = \frac{0.25(3.5)^3}{12}\left(1+6\frac{3}{3.5}\right) = 5.49 \text{ in}^4$$

in which $b = 3.5$ in is the horizontal width of the switch rail and $d = 3$ is the length of the two vertical sides of the switch-rail cross section. Then, assuming the ends of the 54" long switch-rail beam are essentially clamped, the maximum bending stress is

$$\sigma_{bending} = \frac{Pl}{8}\frac{c}{I} = P\frac{54}{8}\frac{3.5/2}{5.49} = 2.15P$$

The maximum deflection of the beam is

$$\Delta_{\max} = \frac{Pl^3}{192EI} = P\frac{54^3}{192(29.5)(10)^6(5.49)} = 5(10)^{-6}(P \text{ in lb}) \text{ in}$$

If we assume $P \le 200$ lb, this stress and deflection are too small to be of concern.
At the attachments, the area of the welds will have to be at least

$$P/\sigma_{design} > 200/10{,}000 = 0.02 \text{in}^2$$

whereas the actual weld area will more than exceed 1 in^2. Hence the switch rails are sufficiently stiff and strong.