10 Guideway Structures

10.1 Introduction

In transit systems that use exclusive guideways, the guideway is generally the largest cost item. Understanding of principles of guideway cost minimization is therefore crucial to the design of economical systems. Exclusive guideways may be either at grade, underground, or elevated; however, the analysis of this chapter is directed only to elevated systems. In spite of generally lower cost, at grade systems are not usually desirable in urban areas because of interference with cross traffic and increased difficulty to clear ice and snow. The cost per unit length of underground systems is roughly proportional to the cross-section area of the tunnel, and hence to the cross-section area of the vehicles. Because of the cost of relocating utilities, underground systems have been estimated generally to be three to five times as expensive per unit length as elevated systems; however, a study performed in Australia[1] indicates that for small-vehicle systems the cost of underground systems may compare favorably with the cost of elevated systems.

The material in this chapter is not intended to provide information needed for detailed design of elevated guideway systems. That would be a lengthier task than can be undertaken in a systems textbook. The objective is rather to provide insight into principles of cost minimization. Detailed methods of dynamic analysis, using computer simulations, have been developed under the auspices of the American Iron and Steel Institute (AISI)[2] and by several university groups[3,4,5,6]. The work of AISI, which includes a comprehensive treatment of ride comfort criteria, may be the most complete modern treatment of the design of steel guideways. The work of Snyder, Wormley, and Richardson[3] is directly useful, not only because they develop methodology for dynamic analysis of guideway-vehicle interactions, but because they give results that permit comparison of required guideway weight per unit length at different vehicle weights. The work of Paulson, Silver, and Belytschko[4] applies most directly to heavy-rail structures. Likins and his colleagues[5,6] include vehicle dynamics as well as guideway dynamics, as do Snyder et al., and also provide a method for minimizing the cost per unit length of a guideway, but for fixed cross-sectional configuration and fixed vehicle speed.

In the analysis of this chapter, it is assumed that the guideway cross section is rectangular for two reasons: (1) it is a basic cross section from which certain specific conclusions can be drawn; and (2) it is sufficiently simply mathematically that the results can be understood in a general context. Various types of loading are considered to determine which loading conditions determine the beam design, and the parameter choices that minimize the beam weight per unit length and therefore its cost, are found.

10.2 Optimum Cross Section Based on Bending Stress

Relationship between Cross-Section Area and Moment of Inertia

Consider a beam of rectangular cross section with the dimensions shown in figure 10-1. If the maximum bending moment on the cross section is M and the maximum bending stress is σ, then it is well known from the theory of strength of materials that

$$\sigma = \frac{Mc}{I} \tag{10.2.1}$$

in which $c = h/2$ and I is the moment of inertia of the cross section. For the cross section of figure 10-1,

$$\begin{aligned} I &= 4t_1\int_0^{(1/2)h} x^2dx + 2w\int_{(1/2)h-t_2}^{(1/2)h} x^2dx \\ &= \frac{t_1h^3}{6} + \frac{wt_2}{2}\,(h^2 - 2ht_2 + 4/3t_2^2) \end{aligned}$$

Substituting into equation (10.2.1),

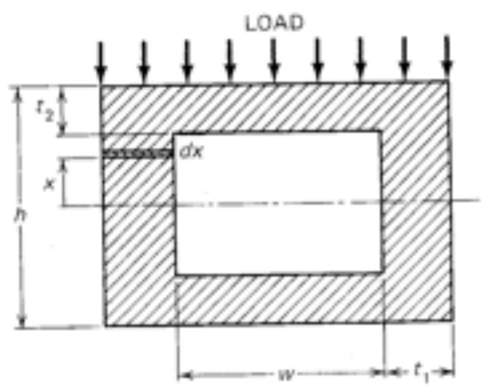

Figure 10-1. Cross Section of a Rectangular Beam

$$\frac{M}{\sigma} = \frac{I}{c} = \frac{t_1 h^2}{3} + \frac{wt_2}{h}\,(h^2 - 2ht_2 + 4/3t_2^2) \qquad (10.2.2)$$

It is convenient to introduce the dimensionless variables

$$\begin{aligned} \mathscr{h} &= h/t_2 \\ \alpha &= wt_2/ht_1 \\ \mathscr{I} &= I/ct_1t_2^2 \end{aligned} \qquad (10.2.3)$$

Then, equation (10.2.2) can be written

$$\mathscr{I} = \mathscr{h}^2/3 + \alpha(\mathscr{h}^2 - 2\mathscr{h} + 4/3) \qquad (10.2.4)$$

The cross sectional area of the beam is, from figure 10-1,

$$A = 2(ht_1 + wt_2) \qquad (10.2.5)$$

or in dimensionless form

$$\mathscr{A} = A/t_1t_2 = 2(1 + \alpha)\mathscr{h} \qquad (10.2.6)$$

By eliminating $\mathscr{h}$ between equations (10.2.4) and (10.2.6), it is possible to obtain an equation for $\mathscr{A}$ as a function of $\mathscr{I}$ with α as a parameter. Then, for a given $\mathscr{I}$, it is possible to determine for what value of α $\mathscr{A}$ will be a minimum. If $\mathscr{A}$ is a minimum, then for given wall thicknesses t_1 and t_2, A is a minimum, and the cost per unit length is a minimum.

First solve equation (10.2.4) for $\mathscr{h}$:

$$\mathscr{h} = \frac{3\alpha \pm \sqrt{(1 + 3\alpha)\left[3\mathscr{I} - \alpha\left(\dfrac{4 + 3\alpha}{1 + 3\alpha}\right)\right]}}{1 + 3\alpha}$$

Only the positive sign in the above equation has physical meaning. This may be seen by noting that for fixed α and fixed t_1 and t_2, h must increase as M increases. Substituting into equation (10.2.6) then leads to the result

$$\mathscr{A} = 2\left(\frac{1 + \alpha}{1 + 3\alpha}\right)\left\{3\alpha + \sqrt{(1 + 3\alpha)\left[3\mathscr{I} - \alpha\left(\frac{4 + 3\alpha}{1 + 3\alpha}\right)\right]}\right\} \qquad (10.2.7)$$

In figure 10-2, $\mathcal{A}/\mathcal{I}^{1/2}$ is plotted from equation (10.2.7) as a function of α with $\mathcal{I}$ as a parameter. Using $\mathcal{A}/\mathcal{I}^{1/2}$ as the ordinate reduces the range of the plotted variable by many orders of magnitude without reducing the generality of the results.

Optimum Width/Depth Ratio

Note from figure 10-2 that there is a value of α that minimizes $\mathcal{A}$ for fixed $\mathcal{I}$, that is, for fixed load and wall thicknesses there is a value of w/h which minimizes the cross sectional area and hence the cost of the beam per unit length. As the wall thickness becomes thin, $\mathcal{I}$ becomes very large. In the limit for very large $\mathcal{I}$, equation (10.2.7) simplifies to

$$\mathcal{A} = \frac{2(1 + \alpha)(3\mathcal{I})^{1/2}}{(1 + 3\alpha)^{1/2}} \tag{10.2.8}$$

Setting the derivative with respect to α equal to zero in equation (10.2.8) gives

$$\frac{\partial \mathcal{A}}{\partial \alpha} = 0 = \frac{2(3\mathcal{I})^{1/2}}{1 + 3\alpha}\left[(1 + 3\alpha)^{1/2} - \frac{3(1 + \alpha)}{2(1 + 3\alpha)^{1/2}}\right]$$

which is satisfied if $\alpha = 1/3$. Substituting $\alpha = 1/3$ into equation (10.2.8) shows that for thin-walled box beams the minimum cross-sectional area is found from

$$\mathcal{A}_{\min} = 4(2/3)^{1/2}\,\mathcal{I}^{1/2} = 3.27\,\mathcal{I}^{1/2} \tag{10.2.9}$$

~~From figure 10-2,~~ It is seen that as α increases, the ratio of $\mathcal{A}$ to $\mathcal{A}_{\min}$ increases as follows:

α:	1	2	3	4	5	6
$\mathcal{A}/\mathcal{A}_{\min}$:	1.06	1.20	1.34	1.47	1.59	1.70

The increased material cost of the beam is in proportion to these numbers.

Figure 10-2 shows that for decreasing $\mathcal{I}$ (thicker-walled beams for a given load), the point of minimum $\mathcal{A}$ moves to values of α smaller than one third, and that for a value of $\mathcal{I}$ between 10 and $10^{1.5}$, α for minimum $\mathcal{A}$ vanishes. It is also noted that for thick-walled beams, the ratio $\mathcal{A}/\mathcal{A}_{\min}$ increases more rapidly as α increases than for thin-walled beams.

Required Wall Thickness

It is of interest to determine if there is an optimum way t_1 and t_2 can be chosen. To obtain a sufficiently high vibrational frequency (section 10.3), it is necessary to choose the ratio of h to the span sufficiently large, and in that way h is determined. Then, for a given α, the beam will support greater load if its wall thickness is greater. Thus the optimum wall thickness is the minimum value that will support the load.

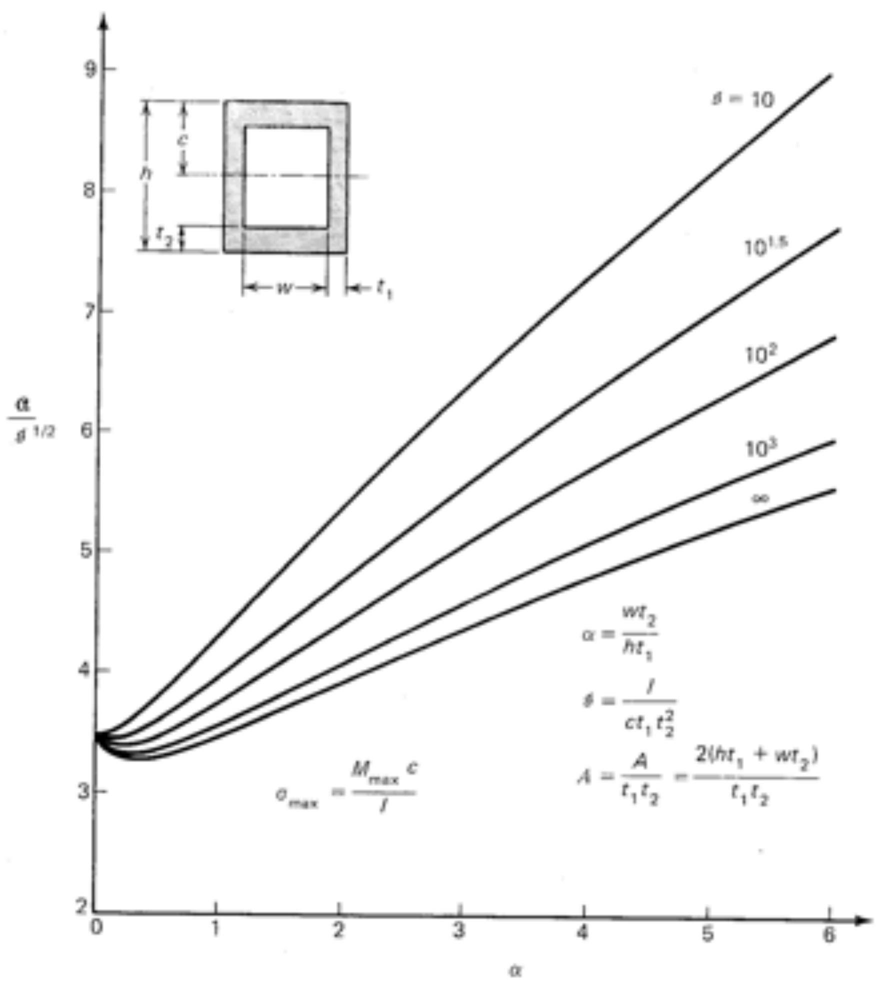

Figure 10-2. Required Cross-Sectional Area of a Box Beam at Given Maximum Static Load

If the minimum wall thickness is thin ($t << h$), equation (10.2.9) applies. Substituting the meaning of $\mathscr{A}$ and $\mathscr{I}$ given in figure 10-2,

$$A_{min} = 3.27\left(\frac{M_{max}}{\sigma_{max}}\right)^{1/2} t_1^{1/2} \qquad (10.2.10)$$

Thus A_{min} depends only on t_1 and not on t_2. This means that t_1 should be chosen as small as practical from the standpoints of fabrication, plate buckling (section 10.7) and plate vibration (section 10.8). On other hand, t_2 can be chosen to accommodate a desired ratio of w/h. Thus, from equations (10.2.3), for $\alpha = 1/3$,

$$\frac{t_2}{t_1} = \frac{h}{3w}$$

If, for ease of material procurement it is desired to make $t_1 = t_2$, then one must choose $h/w = 3$ to minimize cost per unit length. If, however, for some design reason it is desired to choose $h/w = 1$, say, then it is necessary to choose $t_2 = t_1/3$. But we have already chosen t_1 as thin as possible. Therefore t_2 must be at least as large as t_1, but clearly should be no larger. Thus, for thin walled box beams, one should make the choices

$$t_1 = t_2 = t$$

$$h = 3w$$

It is seen that if minimum guideway cost is desired, it is necessary to accommodate the vehicle design to the guideway and not vice versa.

The wall thickness required to meet static stress requirements can be found for $\alpha = 1/3$ by eliminating $\mathscr{A}$ between equations (10.2.9) and (10.2.6). Then, using the parameter definitions given in figure 10-2 and by equations (10.2.3), and setting $\alpha = 1/3$,

$$t = \frac{3M_{max}}{2\sigma_{max}h^2} \qquad (10.2.11)$$

An Example of Optimum Design of a Thin-Walled Beam

Consider a specific example. Assume a uniformly loaded simply supported beam with distance ℓ_s between supports. This load condition represents the case of a span loaded with vehicles with zero spacing between them, the worst static condition that must be considered. Then, from any text on strength of materials,

$$M_{\max} = \frac{q\ell_s^2}{8} \tag{10.2.12}$$

in which q is the load per unit length. If q_ℓ is the live load,

$$q = q_\ell + \rho g A \tag{10.2.13}$$

in which ρg is the weight per unit ~~length~~ volume of the beam.
Substituting for A from equation (10.2.6) with $\alpha = 1/3$

$$q = q_\ell + 8/3\rho g h t$$

Substituting q into equation (10.2.12) and the result into equation (10.2.11) gives

$$t = \frac{3\ell_s^2}{16\sigma_m h^2}\,(q_\ell + 8/3\rho g h t)$$

Solving for t, we have

$$t = \frac{\dfrac{3q_\ell}{8\rho g h}}{\dfrac{2\sigma_m h}{\rho g \ell_s^2} - 1} \tag{10.2.14}$$

Vanishing of the denominator gives the span length ℓ_s for which the beam can no longer support its own weight. Setting the denominator equal to zero and solving for ℓ_s gives

$$(\ell_s)_{\max} = \left(\frac{2\sigma_m h}{\rho g}\right)^{1/2} \tag{10.2.15}$$

We see that the maximum length for $q_\ell > 0$ depends on the material property $\sigma_m/\rho g$. For ordinary structural steel, the yield point is between 30 and 40,000 psi[7]. Therefore, assume a design stress $\sigma_m = 20{,}000$ psi $[140(10)^6\ \mathrm{N/m^2}]$,[a] and $\rho = 484\ \mathrm{lb_m/ft^3}\,[7760\ \mathrm{kg/m^3}]$. Then $\sigma_m/\rho g = 1804$ m.

[a] 1 psi = 6895 N/m², 1 $\mathrm{lb_m/ft^3} = 16.02\ \mathrm{kg/m^3}$, 1 $\mathrm{lb_f/ft} = 14.6$ N/m.

Thus, for a steel beam say 1 m deep, $(\ell_s)_{max} = 60$ m. For reinforced concrete beams, different values will be obtained depending on the arrangement of reinforcing bars and the degree of prestressing.

For a typical live loading $q_\ell = 300$ lb/ft (4470 N/m), the required thickness of a steel beam, computed from equation (10.2.14), is given in figure 10-3. The ratio of live load to dead load at maximum stress

$$\frac{q_\ell}{\rho g A} = \frac{2\sigma_m h}{\rho g \ell_s^2} - 1 \qquad (10.2.16)$$

is also shown as a matter of interest. Note that t is proportional to q_ℓ, but that $q_\ell/\rho g A$ is independent of q_ℓ.

Since $t/h << 1$ in all cases in figure 10-3, the calculation based on equation (10.2.14) is valid. For concrete, t will be much larger and the

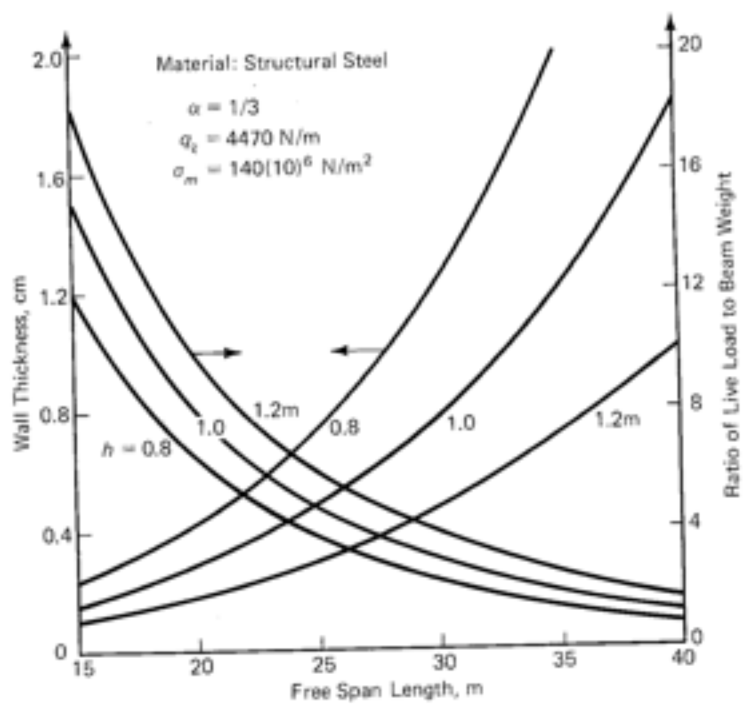

Figure 10-3. Required Wall Thickness of an Optimum Cross Section Steel Box Beam Uniformity Loaded

assumption $\mathscr{I} = \infty$ in use of figure 10-2 is not valid. In this case t can be found by iteration. For a given q_ℓ, $\sigma_{\max}$, and h, assume a value of t and compute $M_{\max}/\sigma_{\max}$. Then compute $\mathscr{I}$ and from figure 10-2 select an appropriate α. Then $\mathscr{A}$ is determined. From equation (10.2.6) t can now be found. If the assumed and computed values do not agree, pick another value of t closer to the computed value and repeat the calculation until the two values converge.

Relationship between Live Load and Weight per Unit Length

An important consideration in guideway design is to understand how the required beam weight per unit length and hence cost varies with q_ℓ. If α and h are held fixed, equation (10.2.6) shows that A is proportional to t. But, from equation (10.2.14) t is proportional to q_ℓ for $t << h$. Thus, for thin-walled beams, A, and hence the beam weight per unit length, increases in direct proportion to q_ℓ. Consequently the beam cost increases with q_ℓ. For thick-walled box beams strict proportionality does not hold and the function $A(q_\ell)$ must be found by iteration between equations (10.2.7) and (10.2.6), in which in $\mathscr{I}$ (equation (10.2.3)) M/σ is substituted for I/c, then M from equation (10.2.12) and q from equation (10.2.13).

Horizontal Wind Loading

Consider horizontal wind loading. From aerodynamic theory the horizontal wind pressure is $\frac{1}{2}\rho_a V_w^2$, in which ρ_a is the air density and V_w is the wind speed. Therefore the wind loading per unit length on the guideway alone with no vehicles is

$$q_{\text{wind}} = \tfrac{1}{2}\rho_a V_w^2 h \, C_D \quad , \; C_D = 1 \tag{10.2.17}$$

The tolerable wind loading on a structure calculated for vertical loading is found with the help of figure 10-2 by noting that the appropriate value of α is the reciprocal of the value used in calculating the beam for vertical loading.

As an illustration, assume t is much less than h and $\alpha = 1/3$ for vertical load. Then, for horizontal wind loads $\alpha = 3$ and, from figure 10-2,

$$\mathscr{A} = 4.38\mathscr{I}^{1/2}.$$

Compare this equation with equation (10.2.9), in which $\mathscr{A}$ has the same value. $\mathscr{I}$ is proportional to M and hence to the load per unit length. Thus

$$q_{\text{wind}} = \left(\frac{3.27}{4.38}\right)^2 (q_\ell + \rho g A)$$

$$= 0.557 q_\ell \left(1 + \frac{\rho g A}{q_\ell}\right)$$

Using the example of figure 10-3, $q_\ell = 4470$ N/m; and as an illustration assume $\ell_s = 30$ m. Then $q_\ell/\rho g A = 3.0$ and the tolerable wind load is

$$q_{\text{wind}} = 3320 \text{ N/m}$$

Substituting this value into equation (10.2.17), the wind speed corresponding to $q_{\text{wind}} = 3320$ N/m is

$$(V_w)_{\max} = \left[\frac{2(3320)}{\rho_a h}\right]^{1/2}$$

But $\rho_a = 1.293$ kg/m^3 at standard conditions, and, in the above example, $h =$ 1 m. Thus

$$(V_w)_{\max} = 72 \text{ m/s} = 161 \text{ mi/h}$$

With winds of even half this magnitude, it can be assumed that the system will be shut down and the vehicles stored in sheltered locations. Thus the added wind load on the vehicles need not be included. With, say $1/3(V_w)_{\max}$, the system may, however, be operative. The wind load on the guideway alone will then be one ninth as much but the torsional load applied to the guideway through the vehicles must be taken into account. This problem is considered in section 10.6.

Double Guideway

Many guideway transit systems, both in development and in operation, use conventional wheeled vehicles which require wide guideways. Based on the above theory, an approach to optimum design of such a structure would be to use two parallel beams rigidly connected together. These could be I-beams, box beams, or some other shape. Let us compare these designs with a single box beam. The variables are the depth of the beams h, the thickness of the material t, and the span ℓ_s. The equations needed to compare designs are equations (10.2.2), (10.2.5), and (10.2.12). For simplicity, assume t is much less than h, and let $\alpha = 1/3$ for each beam. Then, from equations (10.2.2) and (10.2.5),

$$\frac{M}{\sigma} \approx \frac{2}{3}\, th^2 \tag{10.2.18}$$

$$A = \frac{8th}{3} \tag{10.2.19}$$

With the two-beam configuration, M is cut in half for each beam, and we wish to examine the effect of this reduction on the total cross-sectional area, that is, $2A$ for the two-beam configuration. Consider the following three cases.

Case 1: Fixed ℓ_s, t. Then h becomes $h/\sqrt{2}$ and A becomes $A/\sqrt{2}$. Hence the total cross-sectional area A_T becomes $\sqrt{2}A$, where A is the cross-sectional area of a single beam. Thus, with fixed ℓ_s and t, a two-beam configuration has 1.4 times the cross-sectional area of a one-beam configuration. Thus, if the material cost is proportional to the cross-sectional area, the two-beam configuration is 40 percent more expensive for the beams alone; however, the extra labor and material required to fasten the two beams together will increase the cost even more.

Case 2: Fixed ℓ_s, h. Now t becomes ½t, and A becomes ½A. Thus A_T remains the same. But if, for the single beam t is chosen as thin as possible for reasons of fabrication, it is unlikely that it is possible to reduce t by 50 percent. Thus, this form of the two-beam configuration is also more expensive than a single beam.

Case 3: Fixed t, A_T. In this case, equation (10.2.19) shows that h becomes ½h, and equation (10.2.18) shows that M becomes ¼M. Thus two beams can carry only half the moment they must carry. But, from equation (10.2.12), if the maximum moment carried by the two-beam configuration is cut in half, ℓ_s must be reduced by $1/\sqrt{2} = 0.707$, that is, the span length must be reduced by 30 percent, thus requiring 30 percent more support posts.

In all three cases, it is seen that a guideway cost penalty is paid if conventional wheeled vehicles are to be used. Thus, long term interest in monobeam transit systems is justified. The difficult problem, however, has been to design the vehicle/guideway system in such a way that the vehicle can switch from one guideway to another with no moving parts in the track, that is, by means of in-vehicle switches. Several groups, reported in the Lea Transit Compendium[8], have succeeded in developing such switches.

10.3 Dynamic Loading—Single Vehicle Crossing a Span

In design of a guideway of minimum weight per unit length, it is necessary to understand the effect of motion of the vehicles on the maximum stresses in the guideway, and the vertical accelerations produced on the vehicle as a result of motion of the guideway. Also, knowledge of the amplitudes and frequencies of motion of the guideway is needed to make certain that the fatigue life of the guideway will be adequate. The objective of this section is to give some insight into these problems and the parameters that control them.

In this section, the simplest dynamic loading problem of interest is solved and discussed—that of a single vehicle crossing a flexible span. For mathematical simplicity, the dynamics of the vehicle are not taken into account. This permits concentration on guideway characteristics and will provide insight into choice of vehicle dynamic characteristics, but of course in a complete solution, vehicle dynamics must be considered. Such a treatment is given by Snyder, Wormley, and Richardson of MIT[3] for multiple vehicle crossings of a span. Therefore the important multiple vehicle case is treated in section 10.4 by discussion of their computer solutions.

Equation of Motion of a Flexible Span

Figure 10-4 depicts a vehicle of weight W and speed V about to cross a flexible simply supported span of length ℓ_s.

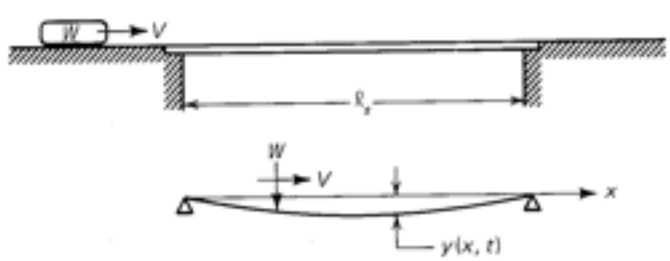

Figure 10-4. A Vehicle Cross a Flexible Span

In the present analysis, the vehicle will be treated as a point force W moving at speed V, and, for mathematical simplicity, the beam will be assumed to be undamped. Since the damping of a real beam is generally small, the undamped beam is a useful idealization. The deflection of the beam, $y(x, t)$, satisfies the partial differential equation[9]

$$EI \frac{\partial^4 y}{\partial x^4} + \rho A \frac{\partial^2 y}{\partial t^2} = f(x, t) \tag{10.3.1}$$

in which E is the modules of elasticity, I and A have the meanings of the previous section, ρ is the mass per unit volume, and $f(x,t)$ is an arbitrary time-space-dependent force per unit length.

For a simply supported beam, the boundary conditions are

$$\left.\begin{aligned} y(0, t) &= y(\ell_s, t) = 0 \\ \frac{\partial^2 y}{\partial x^2}(0, t) &= \frac{\partial^2 y}{\partial x^2}(\ell_s, t) = 0 \end{aligned}\right\} \tag{10.3.2}$$

and the initial conditions are taken as an undeflected beam at rest, that is

$$y(x, 0) = \frac{\partial y}{\partial t}(x, 0) = 0 \tag{10.3.3}$$

Following reference[3], the solution of equation (10.3.1) can be expressed in the form

$$y(x, t) = \sum_{m=1}^{\infty} A_m(t) \sin\left(\frac{m\pi x}{\ell_s}\right) \qquad m = 1,2, \ldots \tag{10.3.4}$$

The form of the space function in equation (10.3.4) satisfies both the differential equation and the boundary conditions (10.3.2). Substitute equation (10.3.4) into equation (10.3.1), multiply by $\sin(n\pi x/\ell_s)$, and integrate from $x = 0$ to $x = \ell_s$. The result is that $A_m(t)$ satisfies the differential equation

$$\ddot{A}_m + \omega_m^2 A_m = \frac{2}{\rho A \ell_s} \int_0^{\ell_s} f(x, t) \sin m\pi x/\ell_s \, dx \tag{10.3.5}$$

in which the dots denote time differentiation, and

$$2\pi f_k = \omega_m = \left(\frac{m\pi}{\ell_s} \right)^2 \sqrt{\frac{EI}{\rho A}} \tag{10.3.6}$$

f_k = natural resonant frequency.

is 2π times the natural frequencies of vibration of the beam.

Solution with Vehicle as Point Load

The assumption that the vehicle behaves as a point force means that

$$\left.\begin{aligned} f(x,t)dx &= W && \text{if} \quad x = Vt < \ell_s \\ &= 0 && \text{if} \quad x \neq Vt \\ &= 0 && \text{if} \quad t > \ell_s/V \end{aligned}\right\} \tag{10.3.7}$$

Substituting equation (10.3.7) into equation (10.3.5) gives

$$\left.\begin{aligned} \ddot{A}_m + \overset{(\omega)}{w}{}_m^2 A_m &= \frac{2W}{\rho A \ell_s} \sin \Omega_m t && t \leq \ell_s/V \\ &= \quad 0 && t > \ell_s/V \end{aligned}\right\} \tag{10.3.8}$$

in which

$$\Omega_m = \frac{m\pi V}{\ell_s} \tag{10.3.9}$$

The general solution of equation (10.3.8), for $t \leq \ell_s/V$, is

$$A_m(t) = C_1 \sin \overset{\omega}{\Omega}_m t + C_2 \cos\omega_m t + \quad \frac{2W \sin\Omega_m t}{\rho A \ell_s(\omega_m^2 - \Omega_m^2)}$$

From equation (10.3.3), the initial conditions are $A_m(0) = \dot{A}_m(0) = 0$. Using these conditions to evaluate the constants C_1 and C_2,

$$A_m(t) = \frac{2W}{\rho A \ell_s \omega_m^2} \frac{1}{(1 - \beta_m^2)} \quad (\sin \beta_m \omega_m t - \beta_m \sin \omega_m t) \qquad (10.3.10)$$

in which, from equations (10.3.9) and (10.3.6),

$$\beta_m = \frac{\Omega_m}{\omega_m} = \frac{\ell_s V}{m\pi} \sqrt{\frac{\rho A}{EI}} = \frac{1}{2m} \frac{V/\ell_s}{f_1} \qquad (10.3.11)$$

is a dimensionless speed parameter.

$$f_1 = \frac{\pi}{2\ell_s^2}\sqrt{\frac{EI}{\rho A}}$$

Comparison with Static Solution

Equation (10.3.10) applies to the case $t \leq \ell_s/V$. The case $t > \ell_s/V$ will be solved later, but first it is useful to compare the above solution with the static solution for the same beam with a concentrated load at the center. From any text on strength of materials, the static midspan deflection ($\cancel{y}$ x = $\ell_s/2$) is

$$y(\ell_s/2) = \frac{1}{48} \frac{W\ell_s^3}{EI} \qquad (10.3.12)$$

For the dynamically loaded beam, equation (10.3.4) gives for the midspan deflection

$$y(\ell_s/2, t) = A_1(t) - A_3(t) + A_5(t) - \ldots \qquad (10.3.13)$$

Using equation (10.3.6), the dimensional coefficient in equation (10.3.10) is

$$\frac{2W}{\rho A \ell_s \omega_m^2} = \frac{2}{\pi^4} \frac{W\ell_s^3}{EI} \frac{1}{m^4}$$

Therefore, the ratio of dynamic to static deflection at midspan is, from equations (10.3.10), (10.3.12) and (10.3.13),

$$\frac{y(\ell_s/2,\ t)}{y(\ell_s/2)} = \frac{96}{\pi^4} \sum_{m=1,3,\ldots} \frac{(-1)^{(m-1)/2}}{m^4} \left(\frac{\sin \beta_m \omega_m t - \beta_m \sin \omega_m t}{1 - \beta_m^2} \right) \tag{10.3.14}$$

Note that $96/\pi^4 = 0.986 \approx 1$.

Because of the factor m^4, equation (10.3.14) is very nearly given by the first term:

$$\frac{y(\ell_s/2,t)}{y(\ell_s/2)} \approx 0.986 \left(\frac{\sin \beta_1 \omega_1 t - \beta_1 \sin \omega_1 t}{1 - \beta_1^2} \right) \tag{10.3.14a}$$

The vehicle reaches midspan when $t = \ell_s/2V$. At this point, from equations (10.3.9) and (10.3.11), $\beta_m \omega_m t = m\pi/2$. Therefore equation (10.3.14a) becomes

$$\frac{y(\ell_s/2,\ \ell_s/2V)}{y(\ell_s/2)} \approx 0.986 \left(\frac{1 - \beta_1 \sin (\pi/2\beta_1)}{1 - \beta_1^2} \right) \tag{10.3.15}$$

Equation (10.3.15) is plotted in figure 10-5. The maximum deflection is 1.520 times the static value and occurs when $\beta_1 = 0.373$. For higher values of β_1 (higher speed) the midspan deflection decreases with speed because the beam has insufficient time to respond to the presence of the vehicle. Below $\beta_1 = 0.373$ the maximum deflection may be larger or smaller than the static value depending on the phase relationship between the natural motion of the beam and the time of arrival of the vehicle at midspan. As the speed decreases to zero, equation (10.3.15) approaches $96/\pi^4$; however, the infinite series of equation (10.3.14) approaches $\pi^4/96$ when $t = \ell_s/2V$ because of the identity

$$\sum_{m=1,3,\ldots}^{\infty} \frac{1}{m^4} = \frac{\pi^4}{96}$$

The maximum deflection of the beam generally occurs before or after the vehicle reaches midspan. These maxima can be found from equation (10.3.14a) but, from continuity, it can be assumed that they follow the envelope indicated by the dotted line in figure 10-5. The maximum midspan deflection while the vehicle is on the beam will probably be larger than 1.520 times the static value, but with the purpose of studying this problem in mind, enough has been learned without computing it exactly.

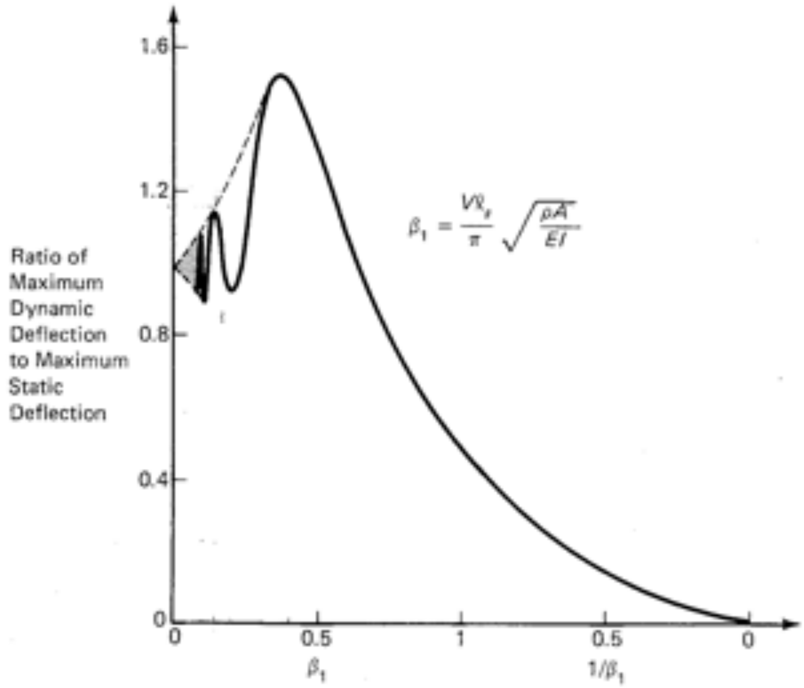

Figure 10-5. Maximum Midspan Deflection of Flexible Beam When Vehicle Is at Midspan

A Critical Speed

The value $\beta_1 = 0.373$ produces the maximum midpoint deflection when the vehicle is at midspan, therefore the corresponding speed can be called a critical speed. From equation (10.3.11) it is

$$V_{cr} = \frac{1.172}{\ell_s}\sqrt{\frac{EI}{\rho A}} \qquad (10.3.16)$$

For a thin-walled steel beam, equations (10.2.2, 10.2.5) show, for the optimum case $\alpha = 1/3$, that

$$\frac{I}{A} \approx \frac{th^3/3}{8th/3} = \frac{h^2}{8} \qquad (10.3.17)$$

Thus, equation (10.3.16) becomes

$$V_{cr} = 0.414\ h/\ell_s \sqrt{\frac{E}{\rho}} \tag{10.3.16a}$$

For steel, $E = 30(10)^6$ psi $= 21(10)^{10}$ N/m^2 and $\rho = 7760$ kg/m^3. Thus

$$V_{cr} = 2150\ h/\ell_s \text{ m/s}$$

and for $h/\ell_s = 1/30$, $V_{cr} = 72$ m/s. This speed is several times the speeds of interest in urban transit applications, but will be of definite interest in designing high-speed intercity systems. For a speed of say 15 m/s, typical of urban applications, $\beta_1 = 0.373(15/72) = 0.078$ with the same set of parameters.

Motion of the Span after Vehicle Has Crossed

We have thus far considered only the deflection of the guideway while the vehile is at its center. It is possible that further motion of the vehicle will add energy to the guideway and hence increase the amplitude of its vibration. Thus, consider the case $t > \ell_s/V$. Then, the right side of equation (10.3.8) is zero and the solution is

$$A_m(t') = \frac{\dot{A}_m(0)}{\omega_m} \sin \omega_m t' + A_m(0) \cos \omega_m t' \tag{10.3.18}$$

in which $t' = t - \ell_s/V$ and $A_m(0)$, $\dot{A}_m(0)$ are found from equation (10.3.10) by substituting $t = \ell_s/V$. Taking into account from equations (10.3.9) and (10.3.11) that $\beta_m \omega_m \ell_s/V = m\pi$, and the expression above equation (10.3.14),

$$\left.\begin{aligned} A_m(0) &= -\frac{2W\ell_s^3\beta_m}{\pi^4 EI} \frac{\sin \omega_m \ell_s/V}{m^4(1-\beta_m^2)} \\ \frac{\dot{A}_m(0)}{\omega_m} &= \frac{2W\ell_s^3\beta_m(\cos m\pi - \cos\omega_m \ell_s/V)}{\pi^4 EI \quad m^4(1-\beta_m^2)} \end{aligned}\right\} \tag{10.3.19}$$

The maximum value of A_m after the vehicle has crossed the span is found by setting $\dot{A}_m(t') = 0$ from equation (10.3.18), solving for $\omega_m t'$, and substituting it into equation (10.3.18). Thus, $\dot{A}_m(t') = 0$ gives

$$\tan\omega_m t' = \frac{\dot{A}_m(0)}{\omega_m A_m(0)} \tag{10.3.20}$$

Using the trigonometric identity $\cos\theta = (1 + \tan^2\theta)^{-1/2}$, equation (10.3.18) can be written in the form

$$A_m(t') = \left[\frac{\dot{A}_m(0)}{\omega_m} \tan \omega_m t' + A_m(0)\right] (1 + \tan^2 \omega_m t')^{-1/2}$$

Substituting equation (10.3.20),

$$(A_m)_{\max} = [A_m^2(0) + \dot{A}_m^2(0)/\omega_m^2]^{1/2} \tag{10.3.21}$$

Substitute equations (10.3.19) into equation (10.3.21), taking into account that $\omega_m \ell_s/V = m\pi/\beta_m$. The result, as a ratio to equation (10.3.12), can be expressed in the form

$$\frac{(A_m)_{\max}}{y(\frac{1}{2}\ell_s)} = \frac{96\sqrt{2}}{\pi^4} \frac{\beta_m}{m^4(1 - \beta_m^2)} (1 - \cos m\pi \cos m\pi/\beta_m)^{1/2} \tag{10.3.22}$$

Substituting into equation (10.3.4) at $x = \ell_s/2$, the ratio of dynamic to static deflection is

$$\frac{y(\frac{1}{2}\ell_s)_{\text{dynamic}}}{y(\frac{1}{2}\ell_s)_{\text{static}}} = \frac{96\sqrt{2}}{\pi^4} \sum_{m-\text{odd}} \frac{(-1)^{(m-1)/2}}{m^4} \frac{\beta_m}{(1 - \beta_m^2)} \left(1 + \cos \frac{m\pi}{\beta_m}\right)^{1/2} \tag{10.3.23}$$

To illustrate the character of the motion, equation (10.3.23) is plotted in figure 10-6. It is seen that the peak amplitude ratio is 1.691 and occurs at $\beta_1 = 0.732$. This compares with a peak amplitude ratio of 1.520 at $\beta_1 = 0.373$ when the vehicle is at midspan. Thus the vehicle-at-midspan critical condition occurs at roughly half the speed, and equation (10.3.16) can still be considered the critical speed.

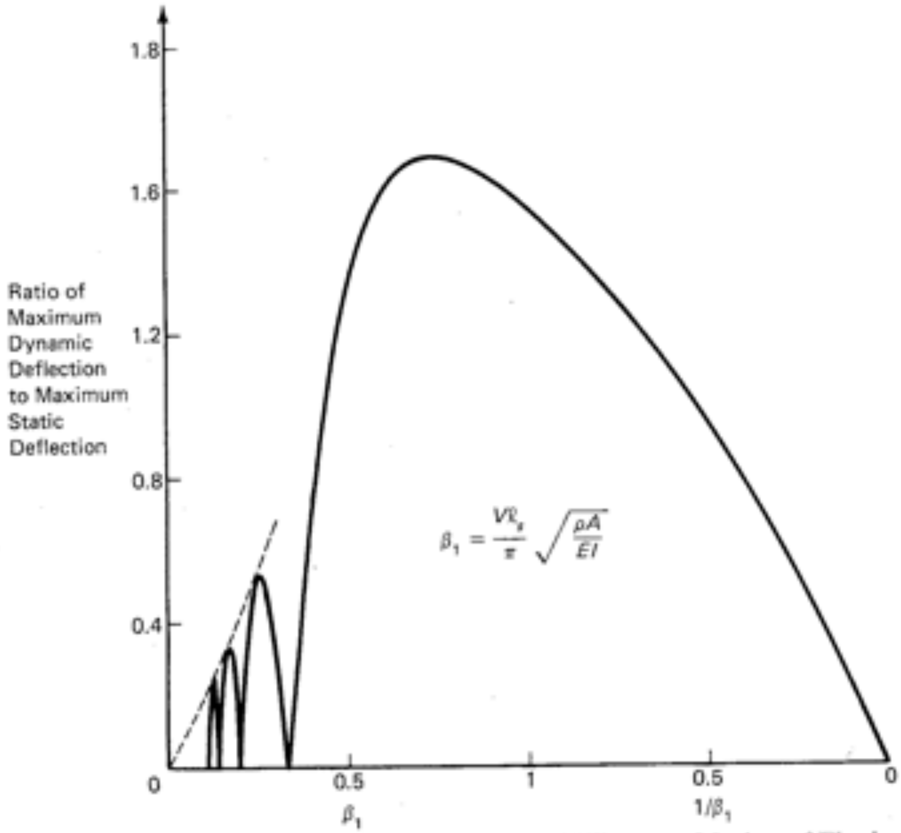

Figure 10-6. Maximum Midspan Amplitude of Vibratory Motion of Flexible Beam After Vehicle Has Crossed It

The Maximum Vertical Acceleration of the Vehicle

The limiting conditions in design of the beam are the maximum acceleration and the maximum dynamic bending stress. Thus formulas for these quantities are now developed.

The vertical acceleration as seen from the moving vehicle is the second total time derivative of $y(x,t)$. Thus

$$\frac{dy}{dt} = \frac{\partial y}{\partial t} + V\,\frac{\partial y}{\partial x}$$

$$\frac{d^2y}{dt^2} = \frac{\partial^2 y}{\partial t^2} + 2V\frac{\partial^2 y}{\partial t \partial x} + V^2\frac{\partial^2 y}{\partial x^2}$$

Applying this operator to equation (10.3.4), then substituting $x = Vt$ and using equation (10.3.9) gives

$$\frac{d^2y}{dt^2} = \sum_{m=1}^{\infty} (\ddot{A}_m \sin \Omega_m t + 2\dot{A}_m \Omega_m \cos \Omega_m t - A_m \Omega_m^2 \sin \Omega_m t)$$

Substitute A_m and its derivatives from equations (10.3.10). Taking into account equations (10.3.9) and (10.3.11), and the identity above equation (10.3.14),

$$\frac{d^2y}{dt^2} = \frac{2W\ell_s V^2}{\pi^2 EI} \sum_{m=1}^{\infty} \frac{1}{m^2(1-\beta_m^2)} \quad [2(\cos^2 \Omega_m t - \sin^2 \Omega_m t)$$

$$+ \left(\beta_m + \frac{1}{\beta_m}\right) \sin \omega_m t \sin \Omega_m t - 2\cos \omega_m t \cos \Omega_m t]$$

Use of trigonometric identities reduces this expression to a sum of single cosine terms

$$\frac{d^2y}{dt^2} = \frac{2W\ell_s V^2}{\pi^2 EI} \sum_{m=1}^{\infty} \frac{1}{m^2}\left[\frac{2\cos 2\Omega_m t}{1-\beta_m^2}\right.$$

$$\left. + \frac{1}{2\beta_m}\left(\frac{1-\beta_m}{1+\beta_m}\right)\cos(\omega_m - \Omega_m)t - \frac{1}{2\beta_m}\left(\frac{1+\beta_m}{1-\beta_m}\right)\cos(\omega_m + \Omega_m)t\right]$$

Note from equation (10.3.11) that β_m varies as $1/m$. Therefore the first term of the above equation is proportional to $1/m^2$, but the second and third terms are proportional to $1/m$. Hence, for $m > 1$ the first term can be neglected in rough estimations. Using equation (10.3.11), factoring out $2\beta_1$, and noting from equation (10.3.16a) that β_1 is small at urban speeds, d^2y/dt^2 can be written in the approximate form

$$\frac{d^2y}{dx^2} = \frac{W}{\rho A\ell_s} \quad \beta_1\left\{4\beta_1 \cos 2\Omega_1 t + \left(\frac{1-\beta_1}{1+\beta_1}\right)\cos(1-\beta_1)\omega_1 t \right.$$
$$- \left(\frac{1+\beta_1}{1-\beta_1}\right)\cos(1+\beta_1)\omega_1 t$$
$$\left. + \sum_{m=2}^{\infty} \frac{1}{m}[\cos(1-\beta_m)\omega_m t - \cos(1+\beta_m)\omega_m t]\right\} \qquad (11.3.25)$$

Since β_1 is much less than 1, the second and third terms of the first vibration mode dominate the above expression. For small β_1, $1/(1 \pm \beta_1) \approx 1 \mp \beta_1$. Then, using trigonometric identities, the second and third terms become approximately

$$2(\sin \Omega_1 t \sin\omega_1 t - \beta_1 \cos\omega_1 t \cos \Omega_1 t)$$

But $\Omega_1 t = \pi/2$ when the vehicle is at midspan, and ω_1 is much greater than Ω_1. Therefore, for β_1 much less than 1, the maximum value of this expression is close to 2. Therefore

$$\frac{1}{g}\left.\frac{d^2y}{dt^2}\right|_{\max} \approx \frac{2W\beta_1}{\rho g A\ell_s} = \frac{2}{\pi}\frac{(W/g)V}{\sqrt{\rho AEI}} \qquad (10.3.26)$$

In which g is the acceleration of gravity.

Minimizing the Maximum Acceleration

Equation (10.3.26) is valid only if β_1 is much less than 1, but from the example following equation (10.3.16a) this is usually true in urban applications. From equation (10.3.11), the condition of small β_1 requires that the material property E/ρ be as large as possible, the cross-sectional property I/A be as large as possible, and for optimized values of E/ρ and I/A, that the span ℓ_s be limited for a given speed, or vice versa. For a given value of β_1, equation (10.3.26) shows that the maximum acceleration depends on $W/\rho g A\ell_s$, the ratio of vehicle weight to span weight. But ρ, A, and ℓ_s have already entered into computation of β_1. Therefore, the acceleration can be held below a specified limit only by limiting the weight of the vehicle. As shown by the rightmost form of equation (10.3.26), the maximum acceleration is insensitive to variations in span length ℓ_s as long as the condition β_1 is much less than 1 is maintained.

As indicated above, the cross-section parameter I/Ashould be maximized to minimize the effects of dynamic loading. Or, for given I/A, the dimension of the cross section should be chosen to minimize A, hence the weight per unit length, and hence cost per unit length. Clearly, the cross section that maximizes I for given A is one in which the bulk of the material is as far from the neutral axis of the beam as possible. For a box beam (see figure 10-1) of given wall thickness, I is maximized if the aspect ratio $\alpha = w/h$ vanishes. This is, of course, an unobtainable condition because lateral stiffness must be provided.

On the other hand, we found in section 10.2 that to maximize the load-carrying ability under static conditions, I/c had to be maximized for a given cross section area and this leads to $\alpha = 1/3$. For this condition, equation (10.2.2) and (10.2.5) show that, for the box beam with thin walls,

$$\frac{I}{A} = \frac{h^2}{12}\left(\frac{1 + 3\alpha}{1 + \alpha}\right) = \frac{h^2}{8}$$

We must concern ourselves with horizontal loads; therefore consider that I/A for the same beam in the horizontal direction is found by interchanging w and h, where $\alpha = w/h$. Thus

$$(I/A)_{\text{horiz}} = \frac{w^2}{12}\left(\frac{1 + 3 \cdot 3}{1 + 3}\right) = \frac{h^2}{8(5.4)}$$

Thus, from equations (10.3.6) and (10.3.11),

$$\frac{(\beta_1)_{\text{vert}}}{(\beta_1)_{\text{horiz}}} = \frac{(\omega_1)_{\text{horiz}}}{(\omega_1)_{\text{vert}}} = (5.4)^{-1/2} = 0.430$$

Thus, the natural frequency of the beam in the horizontal plane is 43 percent of its value in the vertical plane. The load in the horizontal plane on a straight piece of guideway is due mainly to wind, which is a low frequency load. There is also a load due to unbalanced vehicles, much smaller and much more variable than the vertical load.

In curves, the load is due to the centrifugal force WV^2/gR, which for comfort should be less than about $W/4$. Then, from the middle expression in equation (10.3.26), in the horizontal directions $W\beta_1$ becomes

$$\frac{W}{4} \cdot \frac{\beta_1}{0.43} = \frac{W\beta_1}{4(0.43)} = 0.58W\beta_1$$

Thus, without changing ℓ_s, the maximum acceleration in the horizontal plane is only 58 percent of its value in the vertical plane in the case of a box beam for which $\alpha = 1/3$. In conclusion, it appears that a beam aspect ratio α of one-third is large enough to provide adequate stiffness in the horizontal plane.

Weight Penalty for Deviation from Optimum Cross Section

While it is of fundamental importance to find the optimum properties of the cross section for minimum cost per unit length, it is also important to know how much the cost increases if a nonoptimum cross section is used. As before, we assume that the cost per unit length increases with A[b]. Thus, we wish to find how A varies with α for a given value of IA, that is, for given maximum acceleration (see the rightmost form of equation (10.3.26)).

First, substitute $w = \alpha h$ into equation (10.2.5) and solve for h. The result is

$$h = \frac{A}{2t}\frac{1}{(1+\alpha)}$$

Substitute this value of h into the equation for I, which is immediately above equation (10.2.2). Thus

$$IA = A^3\left[\frac{A}{48t^2}\frac{(1+3\alpha)}{(1+\alpha)^3} - \frac{1}{4}\frac{\alpha}{(1+\alpha)^2} + \frac{t^2}{3A}\frac{\alpha}{(1+\alpha)}\right] \quad (10.3.27)$$

[b]Bare material cost will increase in direct proportion to A; however, fabrication costs and auxiliary equipment costs do not increase so rapidly.

The dimensionless parameter A/t^2 must be large in all practical cross sections; therefore, for constant IA, A varies with α for fixed t according to the equation

$$\frac{A}{A_{1/3}} = 1.5\left[\frac{(1+\alpha)^3}{6(1+3\alpha)}\right]^{1/4} \tag{10.3.28}$$

in which $A_{1/3}$ corresponds to $\alpha = 1/3$. Some values of this expression are as follows:

α:	0	1/3	1	2	3	4	6
$A/A_{1/3}$:	0.96	1	1.14	1.34	1.52	1.69	1.98

Comparing with the values following equation (10.2.9), one can see that the cost of the beam increases more rapidly to satisfy the dynamic loading criterion than the bending stress criterion.

Thus, from the viewpoint of vertical dynamic loading only, the cost per unit length is 4 percent less for the ideal and impractical cases $\alpha = 0$, as compared to the bending optimum case $\alpha = 1/3$. If $\alpha = 1/3$ is taken as optimum, it is seen that a square beam costs 14 percent more, the case $\alpha = 3$ costs 52 percent more, and so forth. See footnote on page 280.

The Relationship between Beam Weight and Vehicle Weight at Maximum Acceleration

It is also of importance to understand how the cost of a guideway of a given shape varies with vehicle weight. Thus, for $\alpha = 1/3$ and A/t^2 much greater than 1, equation (10.3.27) shows that

$$IA = \frac{1}{24}\left(\frac{3}{4}\right)^3 \frac{A^4}{t^2}$$

Substituting into equation (10.3.26),

$$\frac{1}{g}\frac{d^2y}{dt^2}\bigg|_{\max} = \frac{4.8(W/g)Vt}{(\rho E)^{1/2}A^2} \tag{10.3.29}$$

Thus, for given maximum acceleration, speed, material; and wall thickness, *the weight of the beam is proportional to the square root of the weight of the vehicles.*

The Relationship between Vehicle Weight and Speed at Maximum Acceleration

From equation (10.2.5), we have for $\alpha = 1/3$, $A = 8th/3$. Substituting this value into equation (10.3.29) gives

$$\frac{1}{g}\frac{d^2y}{dt^2}\bigg|_{max} = \frac{0.68(W/g)V}{(\rho E)^{1/2}th^2} \qquad (10.3.30)$$

As an example, consider the steel beam assumed in computing figure 10-3. For steel, $E = 30(10)^6$ psi $= 21(10)^{10}$ N/m² and $\rho = 7760$ kg/m³; therefore $(E\rho)^{1/2} = 4.0(10)^7$ kg/m²s. Assume $h = 1$ m and take the realtionship between t and ℓ_s from figure 10-3. Then, equation (10.3.30) becomes (t in *cm*)

$$\frac{1}{g}\frac{d^2y}{dt^2}\bigg|_{max} = 1.7(10)^{-6}\ \frac{(W/g)V}{t} \qquad (10.3.31)$$

Note that, because of the direct relationship between t and the span length, ℓ_s, the acceleration decreases as the span increases. From the viewpoint of the fabrication problem, however, there is a minimum practical value of t. Suppose this is one centimeter. Using this value, figure 10-3 shows that the span must be less than 34 m. Assume this is true. Then, consider the tolerable acceleration.

Reference[2], section 4.5, gives standards for vertical vibration recommended by the International Organization for Standardization (ISO). These standards are given as a function of frequency and exposure time. For a transit guideway system the frequency of significance is simply the reciprocal of the time required to traverse a single span, $f_1 = V/\ell_s$. The highest value of f_1 for urban applications may correspond to say $V = 20$m/s, $\ell_s = 20$ m; or $f_1 = 1$ Hz. For frequencies below about $f_1 = 1.4$ Hz, the ISO standard recommends a low-frequency limit vertical acceleration for ride comfort of 0.0707 g. Substituting this value into equation (10.3.31) with $t = 1$ cm gives

$$V = \frac{41600}{M_v} \qquad (10.3.32)$$

in which $M_v = W/g$ in kg if V is in m/s. Thus, for a 1000 kg vehicle, the velocity should not exceed 42 m/s; or for a 2000-kg vehicle, V should not exceed 21 m/s, and so forth. The significance of these results is that, with

1kg = 2.2 lbs

the assumed geometric parameters for a thin-walled steel box beam, stress, not ride comfort, determines the design at urban speeds if only one vehicle passes over each span at one time and the vibrations of the span are not amplified by multiple passages of vehicles. The latter restriction is relaxed in section 10.4.

The Maximum Dynamic Bending Stress

The bending stress in a beam is given by equation (10.2.1) in terms of the bending moment M. From any textbook on strength of materials, M is given in terms of the deflection curve by

$$M = EI\,\frac{\partial^2 y}{\partial x^2}$$

Thus

$$\sigma = cE\,\frac{\partial^2 y}{\partial x^2} \tag{10.3.33}$$

From equation (10.3.4)

$$\frac{\partial^2 y}{\partial x^2} = -\sum_{m=1}^{\infty} A_m(t)\left(\frac{m\pi}{\ell_s}\right)^2 \sin\left(\frac{m\pi x}{\ell_s}\right) \tag{10.3.34}$$

For $t < \ell_s/V$, $A_m(t)$ is given by equation (10.3.10). Thus $\sigma(x,t)$ is found by combining equation (10.3.33) and (10.3.34) and then by substituting equation (10.3.10). Using the relationship above equation (10.3.14), the result is

$$\sigma = \frac{8}{\pi^2}\,\frac{c}{I}\left(\frac{W\ell_s}{4}\right)\sum_{m=1}^{\infty}\frac{(\sin\beta_m\omega_m t - \beta_m \sin\omega_m t)}{m^2(1-\beta_m^2)}\,\sin\left(\frac{m\pi x}{\ell_s}\right) \tag{10.3.35}$$

in which $W\ell_s/4$ is the maximum moment in a statically loaded and simply supported beam with a concentrated load at the center.

In the example used with equation (10.3.16a), it was shown that β_1 is much less than one in urban applications. Taking into account equation (10.3.11), equation (10.3.35) can therefore be approximated by

$$\frac{\sigma}{\sigma_s} = \frac{8}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \sin \Omega_m t \sin\left(\frac{m\pi x}{\ell_s}\right) \tag{10.3.36}$$

This equation applies up to $t = \ell_s/V$. Therefore, from equation (10.3.9), $\Omega_m t$ reaches a maximum of $m\pi$. Consequently, $\sin \Omega_m t$ reaches its maximum of unity for all modes. The first mode ($m = 1$), for example, reaches its maximum when the vehicle is at midspan. Higher modes reach their maxima earlier. Without detailed calculations for a range of values of the parameters, it is not possible to calculate a precise maximum stress; however, it is seen that for the first mode $\sigma_{\max}/\sigma_s = 8/\pi^2 = 0.81$, and for higher modes, the maximum stress falls off as $1/m^2$. Thus, it is unlikely that $(\sigma)_{\max}$ will be much above the static stress, σ_s.

Consider the case $t > \ell_s/V$. From equation (10.3.22), the maximum value of A_m for this case, using equation (10.3.12), is

$$(A_m)_{\max}\Big|_{t>\ell_s/V} = \frac{W\ell_s^3}{EI}\frac{2}{\pi^4}\frac{2\beta_m}{m^4(1-\beta_m^2)}$$

But, from equation (10.3.10), using the expression above equation (10.3.14), the maximum value of A_m when $t < \ell_s/V$ is

$$(A_m)\Big|_{t<\ell_s/V} = \frac{W\ell_s^3}{EI}\frac{2}{\pi^4}\frac{1}{m^4(1-\beta_m)}$$

Hence, after the vehicle passes over the span, the maximum stress does not exceed a factor $2\beta_m/(1 + \beta_m)$ times the maximum stress for $t < \ell_s/V$. For urban speeds, this factor is well under one, therefore the maximum stress is reached while the vehicle is on the span, and, as shown above, is close to the maximum static stress.

During the life of the guideway, it will undergo millions of cycles of stress. In a short-headway system, there may typically be 3000 vehicle passages in the peak hour over a given span, or about 30,000 passages per day. Assuming 300 full days of operation per year, there would be $9(10)^6$ passages per year. Thus, in a fifty-year life time, the guideway would have undergone in the neighborhood of 500 million cycles. It is clear therefore that the maximum stress must be kept well below the fatigue stress limit of the material. From reference[7], p. 5-11, the fatigue stress limit for an indefinite number of stress cycles is given for ordinary structural steel as 30,000 psi ($210 \cdot (10)^6$ N/m^2). Thus, the use of a design working stress limit of 20,000 psi, as has been done in all examples in this chapter, will insure long life of the structure without requiring more expensive special steels.

10.4 Dynamic Loading—Cascade of Vehicles Crossing a Span

Consider a cascade of vehicles crossing the span of figure 10-4 of equal weight W and spaced a distance ℓ_h apart. If each vehicle can be represented by a point load, the integral of equation (10.3.5) becomes

$$\int_0^{\ell_s} f(x,t) \sin m\pi x/\ell_s \, dx = W \sum_{i=1}^{N} \sin m\pi/\ell_s [Vt - (i-1)\ell_h] \tag{10.4.1}$$

in which N is the number of vehicles, but only those terms are included for which

$$0 \leq Vt - (i-1)\ell_h \leq \ell_s \tag{10.4.2}$$

With the help of a trigonometric identity and equation (10.3.9), the right side of equation (10.4.1) may be written in the form

$$W(A \sin \Omega_m t + B \cos \Omega_m t)$$

in which

$$A = \sum_{i=p}^{q} \cos \frac{m\pi\ell_h i}{\ell_s}$$

$$B = \sum_{i=p}^{q} \sin \frac{m\pi\ell_h i}{\ell_s}$$

If $\ell_h > \ell_s$, there is only one vehicle at a time on the span, hence the sums in A and B have only one term. Moreover, between the passage of the first vehicle across $x = \ell_s$ and the arrival of the second vehicle at $x = 0$, the forcing function vanishes. If $\ell_h < \ell_s$, $p = q = 1$ for $0 \leq Vt \leq \ell_h$; $p = 1$, $q = 2$ for $\ell_h \leq Vt \leq 2\ell_h \leq \ell_s$; and so forth. Thus the exact solution must be broken down into time steps corresponding to crossings of the vehicles across the boundaries of the span at $x = 0$ and ℓ_s.

No such solution will be attempted because it is easier to do on a computer, and Synder, Wormley, and Richardson of the department of mechanical engineering at MIT[3] have completed an even more realistic case—one in which each vehicle is represented by a pair of point forces at the front and rear wheels, and in which both structural damping and vehicle dynamics are included. We will discuss that solution, but before doing so, it is useful to study the characteristics of the mutliple vehicle solution in a general way: Motions of each mode m of vibration will be enhanced by successive passes of the vehicles if each vehicle arrives at $x = 0$ at the instant $y_m(x, t) = 0$ and $\dot{y}_m(x, t) > 0$, that is, when the span is just ready to begin its downward motion. If the vehicle arrives at $x = 0$ when $y_m(x, t) = 0$ but $\dot{y}_m(x, t) < 0$, the vehicle's weight will resist the motion of the span, and decrease the amplitude of motion.

The natural frequencies of vibration of the beam are given by equation (10.3.6), and the corresponding periods of motion are $2\pi/\omega_m$. Thus, if $\ell_h/V = 2\pi/\omega_m$ for mode m, that mode will be enhanced. If $f_1 = \omega_1/2\pi$, and equation (10.3.6) is substituted, the critical headways are

$$\frac{\ell_h}{V} = \frac{1}{m^2 f_1} \tag{10.4.3}$$

in which

$$f_1 = \frac{\pi}{2\,\ell_s^2}\sqrt{\frac{EI}{\rho A}} \tag{10.4.4}$$

is the fundamental natural frequency of vibration of the beam if its ends are simply supported.

The quantity ℓ_h/V is the time headway between vehicles. Thus, as the time headway decreases, large amplitude motion can first be expected when the time headway approaches $1/f_1$. Excitation of the second mode can be expected as V/ℓ_h approaches $4f_1$, and so forth. However, since vehicles must operate at a range of spacings down to the minimum permissible, the beam must be designed so that $V/\ell_h < f_1$.

It would appear that

To obtain a feeling for the magnitude of f_1, consider a numerical example based on the numbers used in computing V_{cr} from equation (10.3.16). Thus, using equation (10.3.17) and the value of E/ρ for steel, given after equation (10.3.16a) equation (10.4.4) becomes

$$f_1 = 2890 \ \frac{h}{\ell_s^2} \tag{10.4.4a}$$

where the lengths are in meters. Notice that f_1 is independent of the thickness of the beam walls. If, for example, $h = 1$ m and $\ell_s = 20$ m, $f_1 =$ 7.23 Hz, and $1/f_1 = 0.14$ s. On the other hand, if a longer span, say $\ell_s = 40$ m is desired, $1/f_1 = 0.55$ s.

In section 7.2 it was concluded, based on kinematical considerations, that a minimum headway of the order of 0.25 s is practical if the correct design choices are made. It is now seen that, with a steel beam one meter deep, this appears practical from the structural point of view for simply supported spans of 20 m, but not for spans longer than about 25 m. If the ends are constrained so that they cannot rotate under load, f_1 increases. For completely clamped ends, Timoshenko[9] shows that f_1 increases over the value given by equation (10.4.4) by the factor $(4.730/\pi)^2 = 2.267$. In this case, $1/f_1$ for $\ell_s = 40$ m decreases to 0.24 s. For short-headway transit systems, making the ends of the guideway at the supports rigid may be less expensive than reducing the post spacing or increasing the depth of the beam; however, considering the need for thermal expansion joints, this may be difficult.

An accurate assessment of the minimum practical headway, or of the design parameter choices required for a given headway, requires a detailed computer analysis such as performed by Snyder et al.[3]. Their results pertain to the performance of vehicles travelling on rough guideways as well as on flexible guideways. While the question of tolerable guideway roughness is of crucial importance in the cost of fabrication of the guideway, it is not considered further here. We consider rather the limitations reported by Snyder et al. due to flexible guideways. The results reported there are understood by means of the equation of criticality obtained by setting $m = 1$ in equation (10.4.3). Normalizing with respect to the span length ℓ_s, this equation can be written

$$\frac{\ell_h}{\ell_s} = \frac{V}{f_1 \ell_s} \equiv V_c \quad \leftarrow \text{for max deflection} \tag{10.4.5}$$

in which V_c is referred to by Snyder et al. as the "crossing-velocity frequency ratio.'' It is the ratio of the inverse of the crossing time, ℓ_s/V, that is the crossing frequency, to the fundamental frequency of vibration of the beam. When ℓ_h/ℓ_s, the dimensionless headway, reduces enough to equal V_c, the successive vehicle passages augment the natural vibration of the beam. The computer analysis of Snyder et al. indicates that, as more and more vehicles cross the span, the maximum amplitude of motion continues to build up, and reaches steady state only after the crossing of fifteen to twenty-five vehicles. Also, the maximum deflection of the span is increased after the twenty-fifth vehicle passes by a factor of $3.4/1.8 = 1.9$ over the deflection after one passage. Thus, the maximum beam deflection and stress estimated above for the case of one vehicle crossing are approximately doubled.

Snyder et al. show a series of computer-drawn plots of the nondimensional maximum midspan deflection (y_{max}/y^*), where y^* is the static value given by equation (10.3.12), plotted as a function of V_c (equation (10.4.5)) for four values of ℓ_h/ℓ_s: 1.5, 1.0, 0.5, 0.25. Figure 10-7, taken from Snyder et al.[3] with permission, is a typical example. In this figure, $\ell_s = 30.48$ m and the structural damping ratio of the beam is 0.025. The deflections shown are the steadystate values achieved after fifteen to twenty-five vehicles have crossed the span. The plots can be envisioned as the resulting deflections if a cascade of vehicles at fixed headway ℓ_h continually increases its speed.

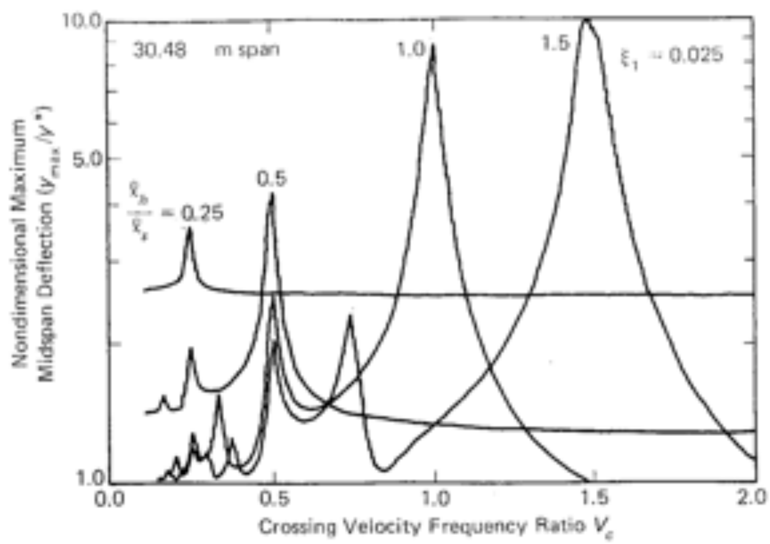

Figure 10-7. Maximum Midspan Deflections due to a Series of 1260-kg Vehicles Crossing a 30.48-m Span (from reference[3], Department of Mechanical Engineering, M.I.T.)

very close to

If $\ell_h/\ell_s = 0.25$, and V is very small, the deflection is the static value for vehicles equally spaced along the span. As V increases, y_{max} doesn't begin to increase noticeably until V_c gets within about 20 percent of the critical value of 0.25 (equation (10.4.5)). At $V_c = 0.25$ the deflection is about 1.4 3,5 times the static value. Then, after the speed has increased so that $V_c \approx 0.30$, the deflection is back to the static value and remains there through the remainder of the range of V_c shown.

If $\ell_h/\ell_s = 0.5$, y_{max} reaches a much higher maximum (6.9 9 times the static value) at the critical value of $V_c = 0.5$, but also peaks at $V_c = 1/4$, and slightly at $V_c = 1/6$. The explanation for this behavior is seen by following the derivation of equation (10.4.3) and concentrating on the fundamental mode, $m = 1$. Thus, if the time headway between vehicles, ℓ_h/V, is equal to the period of motion $1/f_1$, each cycle of vibration of the beam is enhanced by the passage of each successive vehicle. But, the vibratory motion will also be enhanced, though to a lesser extent, if every other or every third, and so forth, vehicle arrives at the beginning of a period of motion of the beam. Thus, successively smaller resonances will occur when

$$\ell_h/V = 2/f_1,\ 3/f_1,\ \ldots$$

Or, in the notation of equation (10.4.5), successively smaller resonances occur when

$$V_c = \frac{\ell_h}{\ell_s},\ \frac{\ell_h}{2\ell_s},\ \frac{\ell_h}{3\ell_s},\ \ldots \qquad (10.4.6)$$

For $\ell_h/\ell_s = 0.5$, $V_c = 1/4$, 1/6, and so forth. For $\ell_h/\ell_s = 1.0$, resonances at $V_c = 1$, 1/2, 1/3, 1/4, 1/5, and 1/6 are visible; and for $\ell_h/\ell_s = 1.5$ resonances may be seen at $V_c = 1.5$, 0.75, 0.5, 0.375, 0.3, and 0.25. Equation (10.4.6) corresponds to equation 5.1 of reference [3] for $m = 1$.

Only the fundamental resonance, corresponding to $V_c = \ell_h/\ell_s$ produces an amplitude above the static value for $\ell_h/\ell_s = 0.25$. On this basis, therefore, the lower-speed resonances need not be avoided. It is important to note also, from figure 10-7, that the peak deflections increase in magnitude as the headway increases. Thus, with long headway systems it is particularly important to avoid operating a long stream of vehicles at $\ell_h = \ell_s V_c$.

Any elevated guideway transit system must be designed for the static loading condition of vehicles end to end on the guideway (equation (10.2.12). Thus, in the case where the minimum operating headway ℓ_h/ℓ_s is 0.25, the worst static condition produces a higher stress if ℓ_v is less than $\ell_h/1.4$, where ℓ_v is the vehicle length. If this condition holds, the resonant condition $V_c\ell_s = \ell_h$ need not be avoided for stress reasons. If the minimum operating headway $\ell_h/\ell_s = 0.5$, we must have $\ell_v < \ell_h/6.9$ if the static condition is to prevail, and so forth. If the system happens to operate with a long stream of vehicles at $\ell_h > (\ell_h)_{min}$, it may, with low probability, operate at one of the secondary resonant points of equation (10.4.6), but these resonances are not strong enough to be of concern. In any case, ride comfort will be increased if long streams of equally spaced vehicles are not permitted to form, that is, random spacing will improve ride comfort.

The MIT group[3] assumed a concrete guideway cross section of fixed shape and, in their computer runs, varied its size until both the stress criterion and the ride comfort criterion were satisfied. They found the required guideway cross sections by making computer runs of multiple vehicle crossings using three vehicle masses with the following characteristics:

Gross Vehicle Mass (kg)	1260	2700	4860
Passenger Capacity	4	6	12
Headway (s)	0.2	0.3	1.0
Flow (persons/s)	20	20	12

The results are shown in figure 10-8. Here we have plotted the required guideway mass per unit length as a function of vehicle mass for the two spans chosen in reference[3]: $\ell_s = 15.2$ m and 30.4 m, and for the cases of a single crossing and of multiple crossings sufficient to produce maximum amplitude of motion. Each of the twelve data points shown corresponds to a set of computer runs required to find the minimum guideway mass per unit length that satisfies both the stress and ride comfort criteria. For instance, the data point corresponding to figure 10-7 is the one for multiple crossings for which $M_v = 1260$ kg and $\ell_s = 30.4$ m. For this case, $f_1 = 1.49$ Hz and $\ell_h = (13.4 \text{ m/s})\ (0.2 \text{ s}) = 2.68$ m. Hence,

$$V_c = \frac{V}{\ell_s f_1} = 0.296 \qquad \text{and} \qquad \frac{\ell_h}{\ell_s} = 0.088$$

In this case, ℓ_h is so small that ℓ_v cannot be much less than ℓ_h. Hence the dynamic load condition is likely to produce the greatest stress. From figure 10-7, it can be inferred, however, that for such a low value of ℓ_h/ℓ_s, the amplitude rise at $V_c = \ell_h/\ell_s$ will be quite small, and that the vehicles have passed over this minor critical point to reach $V_c = 0.296$.

Because of the different relationships to the resonant points in the various data points of figure 10-8 and in other configurations, one must not

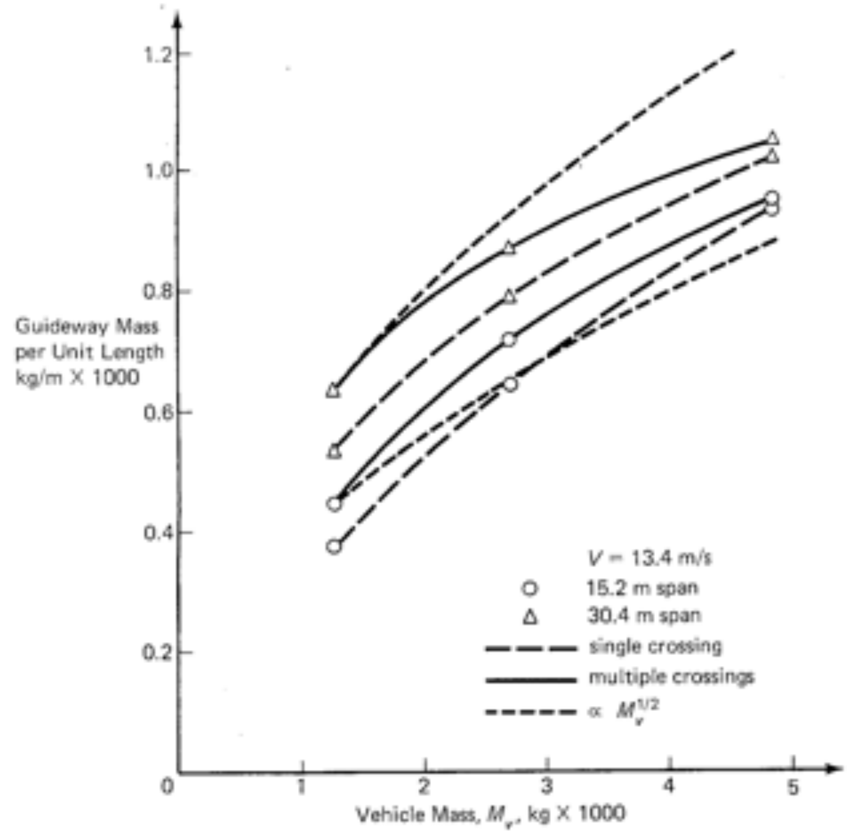

Figure 10-8. Guideway Mass per Unit Length Required to Meet Stress and Ride-Comfort Criteria

generalize the results of figure 10-8 too far; however, we can make the following observations:

1. The required guideway mass increases in going from single crossings to multiple crossings by about 20, 11, and 1.5 percent for vehicles masses of 1260, 2700, and 4860 kg, respectively. Thus, at least in this example, the analysis of multiple crossings is significant only in the smaller-vehicle cases, but the increase in guideway mass is small enough so that the single vehicle crossing, analyzed exactly, is of much interest in understanding the basic phenomena. (It must be remembered, however, that the MIT study also took into account vehicle dynamics.)
2. At a constant flow of persons per second, smaller vehicles will result

in a lower guideway mass per unit length and hence lower cost. Note that the data corresponding to the largest of the three vehicles corresponds to a flow only six tenths of the flow in the two other cases. Thus, if the flow with the largest-mass vehicle were increased to correspond to the other two cases, its guideway would be substantially larger.

From equation (10.3.29) it was concluded that the mass per unit length of the guideway is proportional to the square root of the vehicle mass if the ride comfort criterion governs over the stress criterion. To test this hypothesis in the case of the data of figure 10-8, dashed curves proportional to $M_v^{1/2}$ are drawn from the multiple crossing data points for the lightest vehicle. It is seen that the square-root assumption underestimates the guideway mass when $\ell_s = 15.2$ m, but overestimates it when $\ell_s = 30.4$ m. The differences are not surprising, however, because of: (1) the lower flow for the heaviest vehicle; (2) the fact that equation (10.3.29) is approximate; and (3) the fact that figure 10-8 applies for one specific velocity. The relationship

$$\text{GUIDEWAY MASS} \propto (\text{VEHICLE MASS})^{1/2}$$

is still a good rough approximation.

10.5 Limit Valve of Speed Based on Ride Comfort

Based on the analysis of section 10.4, it is useful to consider the following simplified analysis of ride comfort in vehicles in a cascade: Consider the case where ℓ_h/ℓ_s is much less than 1 and assume the guideway is uniformly loaded with a load per unit length $q = W/\ell_h$, where W is the vehicle weight. Based on figure 10-7, assume that for small ℓ_h/ℓ_s the resonant effects are small and, therefore, that the beam deflection is the static deflection with the load q. To increase ride comfort, assume that the beams are precambered to lie flat on their support posts when no vehicles are present. Thus, the deflection will be totally due to vehicle weight. To increase the beam natural frequency, clamp the ends of the beams on the support posts by overlaying steel sheets secured to the beams in such a way that thermal expansion can take place.

From any text on strength of materials, the maximum moment in such a beam is

$$M_{\max} = \frac{q\ell_s^2}{12} \qquad (10.5.1)$$

and the maximum deflection is

$$\Delta_{\max} = \frac{q\ell_s^4}{384EI} \tag{10.5.2}$$

With clamped beams, we can assume that the vertical displacement seen by a passenger is

$$y(t) = \frac{\Delta_{\max}}{2} \sin \omega t$$

where

$$\omega = 2\pi V/\ell_s$$

Hence, the maximum vertical acceleration is

$$a_m = \frac{\Delta_{\max}}{2} \ \omega^2 = 2\pi^2 \Delta_{\max} V^2/\ell_s^2 \tag{10.5.3}$$

Assume the beam is designed to a certain maximum bending stress σ_m under the total load of vehicles end to end and beam weight. Then, from equations (10.2.1) and (10.5.1), the required moment of inertia is

$$I = \frac{h}{2\sigma_m} M_{\max} = \frac{h\ell_s^2}{24\sigma_m}\left(\frac{W}{\ell_v} + \rho g A\right)$$

in which $h = 2c$, ℓ_v is the vehicle length, and ρgA is the weight per unit length of the beam. Substituting this value of I into equation (10.5.2), and remembering that because of the assumed camber, $q = W/\ell_h$ in computing maximum deflection, equation (10.5.2) becomes

if vehicles

$$\Delta_{\max} = \frac{\sigma_m \ell_s^2 \ell_v}{16Eh\ell_h} \frac{1}{(1 + \rho gA\ell_v/W)}$$

Substituting this expression into equation (10.5.3) and solving for V, we obtain

$$V_{\text{lim}} = \frac{2}{\pi}\left[\frac{2a_m E h \ell_h}{\sigma_m \ell_v}\left(1 + \frac{\rho g A \ell_v}{W}\right)\right]^{1/2} \tag{10.5.4}$$

It is worthwhile to note from structural theory that if the beam had been simply supported, the factor of 12 in equation (10.5.1) would become 8, and the factor of 384 in equation (10.5.2) would become 384/5. Thus, E in equation (10.5.4) should be multiplied by $12/(8 \cdot 5) = 0.3$, and the maximum value of V for adequate ride comfort would reduce by $(0.3)^{1/2} = 0.548$.

We see that increasing the maximum stress σ_m lowers the limit velocity. This is because σ_m permits higher deflection. As expected, a heavier beam (greater $\rho g A$) increases the limit velocity, heavier vehicles lower the limit velocity, and a shorter headway between vehicles lowers the limit velocity. The span ℓ_s does not enter directly, but through the parameter A, which must increase with ℓ_s at a given σ_m. The limit velocity can be increased most easily be increasing h.

As indicated in connection with equation (10.3.32) assume $a_m = 0.707$ m/s². Then for a one-meter-deep steel beam ($E = 21(10)^{10}$ N/m², $\sigma_m = 140(10)^6$ N/m²), equation (10.5.4) becomes

$$V_{\text{lim}} = 29.3\left(\frac{\ell_h}{\ell_v}\right)^{1/2}\left(1 + \frac{\rho g A \ell_v}{W}\right)^{1/2} \text{ m/s} \tag{10.5.5}$$

Thus, without computing the guideway/vehicle weight parameter, it is seen that for all headways the limit value of speed for adequate ride comfort is above the range of speeds of interest for urban applications. On the other hand, if the beams are simply supported, the factor 29.3 reduces to $29.3(0.548) = 16.1$ m/s = 36 mi/hr. In this case the parameter $\rho g A \ell_v/W$ is of interest for heavy vehicles at short headways. For steel, $\rho g = 77600$ N/m³. Assuming an optimum steel beam, equation (10.2.5) shows that $A = 8ht/3$. Assume $t = 1$ cm, $h = 1$ m. Then $\rho g A = 2069$ N/m. As an example, take the middle vehicle of figure 10-8. Then $W = 2700(10) = 27{,}000$ N. Thus $\rho g A/W = 0.077$ m^{-1}. Consider a minimum time headway of 0.25 second. Then, for $V = 20$ m/s, $\ell_h = 20(0.25) = 5$ m. Assume $\ell_v = 3$ m. Then for this case $\rho g A \ell_v/W = 0.23$ and $V_{\text{lim}} = 23.0$ m/s (52 mi/hr) for simply supported beams. Since this limit speed is above the assumed speed, the beam is still determined by the bending stress criterion and not by ride comfort. With simple supports, however, the ride over the supports will contain higher frequency components than assumed and requires further study. To take into account the amplification produced near resonant conditions, a_m in equation (10.5.4) should be reduced in the same proportion that the amplitude is increased over the static value near resonance. If this adjustment is made to equation (10.5.4), it applies to the dynamic as well as to the static deflection conditions.

10.6 Torsion

Torsional Loads

A transit guideway can be loaded in torsion due either to wind forces or to centrifugal forces. Consider first the wind load. As an extreme condition, assume the maximum torque on the guideway to be determined by the condition that vehicles of height h_v are parked end to end on a span of length ℓ_s. Assume that at the support posts, the guideway is constrained from rotating. Then the torque is greatest at the support posts and is equal to one half the torque produced on an area $h_v\ell_s$. If h is the depth of the guideway, assume the point of application of the wind load is $(h + h_v)/2$ from the axis of twist of the beam. Then the torque due to wind is

$$T_w = \frac{\rho_a V_w^2}{8} \; h_v \ell_s (h + h_v) \tag{10.6.1}$$

in which $\rho_a = 1.293$ kg/m^3 is the air density at standard conditions and V_w is the maximum wind speed perpendicular to the guideway.

A centrifugal torque is produced by vehicles travelling around a curved section of guideway at line speed V_L. If the vehicle weight is W and the maximum comfort level of lateral acceleration is a_ℓ/g, each vehicle produces a maximum torque

$$T_c = W \frac{a_\ell}{g} \frac{(h + h_v)}{2}$$

If the headway is ℓ_h, and ℓ_s is the unsupported span length around a curve, there can under normal conditions be ℓ_s/ℓ_h vehicles on one span. Then, as with equation (10.6.1), the maximum torque is less than

$$(T_c)_{\max} = \frac{W a_\ell}{4g} \; (h + h_v) \; \frac{\ell_s}{\ell_h} \tag{10.6.2}$$

The actual torque is less than that given by equation (10.6.2) because the curvature of the guideway causes part of the torque to add to the bending moment at the support post, and only a component to be a true torque. Hence the present calculation is conservative.

The ratio of wind to centrifugal torque is

$$T_w/T_c = \frac{\rho_a V_w^2 h_v \ell_h}{2W(a_\ell/g)} \frac{(\ell_s)_{\text{straight}}}{(\ell_s)_{\text{curved}}} \tag{10.6.3}$$

Assume the system operates normally up to say $V_w = 60$ mi/h (27 m/s), and consider two cases: (1) Standing passenger vehicles—assume $a_\ell/g = 0.125$ g, $W = 10{,}000$ lb$_f$ (45,400 N), $h_v = 2.4$ m, $\ell_h = 130$ m, and ℓ_s is the same for straight and curved track. Then, from equation (10.6.3),

$$\frac{T_w}{T_c} \approx 26$$

(2) Seated passenger vehicles—assume $a_\ell/g = 0.25$ g, $W = 2000$ lb$_f$ (9080 N), $h_v = 1.6$ m, $\ell_h = 10$ m, and ℓ_s is again the same in both cases. Then

$$\frac{T_w}{T_c} \approx 3.3$$

Thus, in both cases the wind torque is dominant and can be assumed to act with no centrifugal forces in computation of the maximum torsional stress, because it is not prudent to operate at normal line speed when the wind speed is maximum.

The Torsional Stress in a Box-Beam Guideway

Assume the guideway is a box beam with the dimensions shown in figure 10-1. Let $t_1 = t_2 = t << h$ and let the shear stress on the cross section due to the applied torque T be τ. According to Timoshenko and Goodier[10] , Saint-Venant showed that the cross sections of a noncircular beam warp in torsion. If they are restrained from warping, additional stress concentrations occur, and the following analysis must be modified. Thus, near constrained ends, more detailed knowledge of the means by which the supports resist torsion than available here must be available for a rigorous solution. Such a solution has been carried out by Ebner[11] but it will be assumed (1) that for relatively thick-walled beams (compared to aircraft wings) the correction is small, or (2) that it is possible to design end constraints that minimize stress concentration in torsion.

With these caveats, consider the beam of figure 10-1 and assume that the wall is thin compared with the depth h. In this case, Tomoshenko and Goodier indicate that, except at the corners (considered below), the shear stress τ can be assumed uniform. Then, the torque T can be expressed as

$$T = 2\tau t(hw/2 + wh/2) = 2\tau thw$$

Thus,

$$\tau = \frac{T}{2tA_c} \tag{10.6.4}$$

where $A_c = hw$ is the cross-sectional area of the beam.

The material cross-sectional area A is given by equation (10.2.5). The shear stress is minimum for a given guideway weight per unit length if A_c is maximized with A held constant. Thus, setting the variation of A equal to zero with t fixed results in $\delta w = -\delta h$. Then

$$\delta A_c = 0 = w\delta h + h\delta w = (w - h)\delta h$$

Thus, as should have been expected from symmetry, a box-beam guideway will resist a given load with the minimum weight per unit length if the beam is square. This is an important consideration, however, only if the torsional wind load is the dominant factor in determining the size of the guideway.

According to Timoshenko and Goodier, equation (10.6.4) is valid away from the corners of the box beam. At the corners, there is a shear-stress concentration dependent upon the ratio of the inside radius of curvature at the corners, a, to the wall thickness, t. On page 301, reference[10], a curve of the stress concentration as a function of a/t is given, calculated both on the basis of an approximate analytical theory and a numerical calculation by finite differences. For example, for $a/t = 1.0$, $\tau_{max}/\tau = 1.3$; and for $a/t = 0.5$, $\tau_{max}/\tau = 1.7$. To be safe, assume the stress concentration factor to be two, so that, from equation (10.6.4),

$$\tau_{max} = \frac{T}{tA_c} \tag{10.6.5}$$

Substituting equation (10.6.1) into equation (10.6.5), the maximum shear stress ($A_c = hw = \alpha h^2$) is

$$\tau_{max} = \frac{\rho_a V_w^2}{8\alpha} \frac{\ell_s}{t} \frac{h_v}{h} \left(1 + \frac{h_v}{h}\right) \tag{10.6.6}$$

Assume $\rho_a = 1.293$ kg/m^3, $V_w = 27$ m/s, $\alpha = 1/3$, $h = 1$ m, and $h_v = 2.4$ m. Then

$$\tau_{max} = 2884 \frac{\ell_s}{t} \frac{N}{m^2} = 0.410\ell_s/t \text{ psi}$$

Taking values of ℓ_s/t from figure 10-3, the following stresses are found:

$\ell_s(m)$:	20	30	40
τ_{max}(psi):	2830	1585	893
τ_{max}(N/m^2):	19.9(10)^6	11.2(10)^6	6.3 (10)^6

Thus, for a design based on bending stress, the shear stress is greatest where ℓ_s and t are the smallest. Since the design shear stress for structural steel[7] is about 12,000 psi (84(10)^{6}N/m^2), the design is in all cases determined by bending stresses, not shear stresses. If the minimum plate thickness is limited to say 1 cm for ease of fabrication, as assumed following equation (10.3.31), the maximum shear stress is even less for the smaller values of ℓ_s.

This conclusion leads to increased flexibility in design because the full box beam is not needed for resisting torsional loads. As an example, assume $w = 1/3$ m but that, for the portion of the box beam that resists torsion, h is reduced to the value h' such that τ_{max}, in equation (10.6.5), is equal to the design value. Then, substituting, $A_c = wh'$ and equation (10.6.1) for $\boldsymbol{T}$, equation (10.6.5) can be expressed in the form

$$\begin{aligned} h' &= \frac{\rho_a V_w^2}{8\tau_{max}} \frac{\ell_s}{t} \frac{h_v}{w} (h + h_v) \\ &= 17.5(10)^{-6}\ \ell_s/t \text{ for } h_v = 1.6 \text{ m} \\ &= 34.4(10)^{-6}\ell_s/t \text{ for } h_v = 2.4\text{m} \end{aligned} \tag{10.6.7}$$

in which, in computing the lever arm for the wind torque, h is still taken as 1

m. Assume $t_{min} = 1$ cm, as suggested above. Then figure 10-3 shows that t is greater than t_{min} only above ℓ_s about 34 m. With this assumption, several values of h' are as follows:

ℓ_s(m)	t(cm)	h_v(m)	h'(cm)
20	1	1.6	3.50
30	1	1.6	5.25
40	1.84	1.6	3.80
20	1	2.4	6.88
30	1	2.4	10.32
40	1.84	2.4	7.48

Thus, a much shallower beam than a full box beam one meter deep will provide ample torsional strength. Note from equation (10.6.7) that h' is a quadratic function of the height of the vehicle, h_v, but that for full, unslotted box beams, extra vehicle height is not significant in torsion from the viewpoint of shear stress. (It may, however, be a factor in lateral vehicle stability.) If increased h_v adds to the weight of the vehicle; however, it will require a heavier beam. These conclusions indicate that the use of U-shaped beams to simplify switching, as is done in the design developed by The Aerospace Corporation[12] is a practical configuration.

Slotted Box-Beam Guideways

One of the most difficult design problems in narrow-beam transit systems is to develop a practical configuration that can permit vehicles to switch without moving a portion of the track. The requirement of movement of the track generally restricts the systems to long headways, hence large vehicles, hence to a large cross-section, high cost guideway. A method around this problem, employed in the design of the Rohr Monocab system in the United States, and the H-Bahn system in West Germany, is to slot the guideway so that a suspension bogie can ride inside the beam[13]. The subject of this subsection is the analysis of the torsional stresses in such a beam.

The theory of torsion of open channel sections is developed by Timoshenko and Goodier[10].

From this work, the maximum torsional stress in a thin-walled channel section and away from a corner is found to be the same as the maximum torsional stress in a bar of narrow cross section of thickness t and width b if the developed length of the channel cross section is b and its thickness is also t. Thus, reference[10] gives for the maximum shear stress of a channel section the formula

$$\tau = \frac{3T}{bt^2}$$

in which T is the applied torque. Just as with the development of equation (10.6.5), to account for stress concentrations in the corner, we will multiply the above value by two to obtain

$$\tau_{\max} = \frac{6T}{bt^2}$$

For a thin-walled slotted box beam of depth h and width w, $b = 2(h + w)$ if we neglect the width of the slot. Then

$$\tau_{\max} = \frac{3T}{(h + w)t^2} \tag{10.6.8}$$

In this case, the maximum stress is independent of the ratio w/h and depends only on the length of the perimeter.

Comparing with equation (10.6.5) with $A_c = wh$, we see that the slot increases the maximum stress by the factor

$$\frac{(\tau_{\max})_{\text{slot}}}{(\tau_{\max})_{\text{no slot}}} = \frac{3wh}{(h + w)t} \tag{10.6.9}$$

For beams with $h = 1$ m, $w = 1/3$ m, $t = 1$ cm, this ratio is 75. Thus, while the torsional stress in a full box beam was well below the design stress, it will dominate the design of a slotted box beam.

Substitute equation (10.6.1) into equation (10.6.8).
Then

$$\tau_{\max} = \frac{3\rho_a V_w^2}{8} \frac{h_s \ell_s}{t^2} \left(\frac{h + h_s}{h + w} \right) \tag{10.6.10}$$

As in the previous subsection, let $\rho_a = 1.293$ kg/m³ and $V_w = 27$ m/s. Then

$3\rho_a V_w^2/8$ = 353 N/m² = 0.05 psi. Again assume a design shear stress of 12,000 psi or $84(10)^6$ N/m². To get a feeling for magnitudes, let h = 1 m, w = 1/3 m, and ℓ_s = 20 m. Then from equation (10.6.10) the thickness needed to resist the torsional load is

$$t = 1.60 \text{ cm for } h_v = 1.6 \text{ m}$$

$$= 2.24 \text{ cm for } h_v = 2.4 \text{ m}$$

For a closed box beam, it was argued that the walls should be at least 1 cm thick to simplify fabrication. If this proves correct, the penalty in added weight per unit length and hence cost per unit length in using a slotted box beam is a factor of 1.6 for h_v = 1.6 m and 2.24 for h_v = 2.4 m. Recall from the previous subsection that these two values of h_v correspond to the use of seated- or standing-passenger vehicles, respectively. Thus, with slotted box beams, the penalty in guideway cost in using standing-passenger versus seated-passenger vehicles is a factor of 2.24/1.60 = 1.40 or 40 percent.

An alternative design for a slotted beam configuration is to use a shallow box defined by equation (10.6.7) to resist torsion and thinner walls to contain the bogie. The fabrication cost of such a design would, however, increase.

10.7 Plate Buckling

If a box-beam guideway is built up of thin steel plates, it is necessary to ascertain that the side walls are thick enough so that they will not fail by buckling. The theory of buckling is given by Timoshenko[14]. To determine the critical buckling load of the side walls of a box beam, we must define the manner of support of the edges of the side walls and the manner of loading. These factors will differ with different vehicle support configurations, but before troubling to define them in detail it is useful to consider a simplified configuration which can be expected to produce buckling most easily. Then we can compare it with other configurations.

The simplest configuration of interest is a plate of length ℓ, width h, and thickness t simply supported along all its edges and subject to a uniform load per unit length N along the boundaries of length ℓ, in which we assume ℓ is much greater than h. Then, from reference[14], p. 329, the critical buckling load, converted to our notation, is

$$N_{cr} = \frac{\pi^2 D}{\ell^2}\left(\frac{\ell}{h} + \frac{h}{\ell}\right)^2 = \frac{\pi^2 D}{h^2}\left(1 + \frac{h^2}{\ell^2}\right)^2$$

in which

$$D = \frac{Et^3}{12(1 - \nu^2)}$$

is the plate rigidity factor, where ν is Poisson's ratio and E is the modulus of elasticity. For structural steel $\nu = 0.3$ and $E = 21(10)^{10}$ N/m².

Thus, the critical buckling load for a steel plate for which h is much less than ℓ is

$$N_{cr} = 19(10)^4 \frac{t^3}{h^2} \text{N/m} \qquad (10.7.1)$$

in which t is in centimeters and h is in meters. For the plate used in previous calculations for which $t = 1$ cm and $h = 1$ m,

$$N_{cr} = 19(10)^4 \text{ N/m} \quad (12{,}800 \text{ lb}_f/\text{ft})$$

Compare this result with figure 5-4, where a vehicle mass of 1000 kg/m exerts a force of 10,000 N/m. A pair of simply supported plates, idealizing a box beam, could support up to 380,000 N/m or a mass distribution of 38,000 kg/m—far higher than the mass per unit length of any of a wide range of transit vehicles. The actual edge conditions would stiffen the plate and decrease its load, and the load of the vehicles in most practical configurations is not applied directly to the top of the plates. Therefore, plate buckling plays no role in designing box-beam structures of the approximate dimensions considered above.

As a matter of interest, plate buckling would be important if N_{cr} were reduced by a factor of about 40(see figure 5-4). This would occur if t were reduced by $(40)^{1/3} = 3.42$ to about 3 mm. Such thin plates would be difficult to handle in a steel fabrication shop.

10.8 Plate Vibration

A final factor in design of thin-plate beams is the possibility that plate vibration will produce unwanted noise as vehicles pass. This problem can be analyzed by means of the theory of vibration of plates, developed by Timoshenko[9]. His equation (214) gives the frequencies of vibration of rectangular plates with simply supported edges. In the notation used here, these frequencies can be expressed in the form

$$f_{mn} = \frac{\pi}{2} t \sqrt{\frac{E}{12(1 - \nu^2)\rho}} \left(\frac{m^2}{h^2} + \frac{n^2}{\ell^2} \right) \qquad (10.8.1)$$

in which m and n are positive integers. For steel ($E = 21(10)^{10}$ N/m^2, $\rho =$ 7760 kg/m^3, $\nu = 0.3$), equation (10.8.1) becomes

$$f_m = 24.70t \left(\frac{m^2}{h^2} + \frac{n^2}{\ell^2} \right) \text{ Hz}$$

in which t is the thickness of the plate in centimeters and, as before, h and ℓ are the depth and length in meters, respectively.

Thus, for $t = 1$ cm, $h = 1$ m, and ℓ much greater than h, the lowest frequency ($m = n = 1$) is 24.7 Hz. The lowest audible frequency is about 20 Hz. Thus, corresponding to m, $n = 1, 2, 3, \ldots$, sound will be produced throughout the audible range if the beam is excited. Measures to prevent the production of unwanted noise depend on two factors: (1) design of the suspension system of the vehicles in such a way that the vibratory modes of the side walls are not excited; and (2) damping of the vibratory modes by application of an appropriate material to the walls of the plate.

If the vehicles use wheels, for example, the use of steel wheels on steel rails would appear to be the worst combination because imperfections in the rails attached to the guideway and in the wheels would act as forcing functions and would cause the plate walls of the guideway beam to vibrate audibly as the vehicles pass. Use of rubber-tired wheels would dampen the effect of imperfections, and the use of air or magnetic suspension would appear to remove high frequency forcing functions altogether.

The damping of structures by use of surface treatments is discussed by Plunkett[15,16]. For the present application, the work of reference[16] appears directly applicable. There it is shown that the application of a thin viscoelastic layer constrained by even thinner sheets of a stiff material such as steel or aluminum of optimum length provides a significant amount of damping to thin beams even though the surface treatment adds less than 3 percent to the weight of the beam. The optimum constraining layer length for damping of a thin beam is $3.28\ (t_1 t_2 E_2/G_1)^{1/2}$, where E_2 and t_2 are the elastic modulus and thickness of the constraining layer, respectively; and G_1 and t_1 are the shear modulus and thickness of the viscoelastic material, respectively. The amount of damping falls off rapidly if the length of the constraining layer deviates from optimum. Torvik and Strickland[17] have extended application of the method to plates with similar results.

10.9 Optimum Span Length

According to equation (10.2.14), the plate thickness of a box beam of constant depth and width must increase with the span length ℓ_s in a stress-limited design. Thus, the cost of the beam decreases as ℓ_s decreases, but smaller ℓ_s means more support posts; therefore, there is a value of ℓ_s that minimizes the cost of the guideway plus support posts. It is instructive to study this optimum design point.

To be specific, consider a thin-walled box beam design. The cost per unit length of the beam can be expressed as

$$C_g = C_{g0} + C_{kg}\rho 2(h + w)t(\ell_s) \tag{10.9.1}$$

in which C_{g0} is the element of cost per unit length independent of ℓ_s, C_{kg} is the erected cost per kilogram of material, $2(h + w)t$ is the cross-sectional area of the beam, and $t(\ell_s)$ indicates the dependence of t on ℓ_s from equation (10.2.14). Assume, however, that t is not permitted to go below a value t_m taken in previous subsections as one centimeter.

Assume the posts are square cross-sectioned steel beams of side length h_p at the base, of wall thickness, t_p, and of height ℓ above the ground. Let the cost per post be

$$C_p = 4C_{pkg}\rho t_p \ell h_p \tag{10.9.2}$$

in which it is assumed that the total cost of the post counting its base is proportional to the above-ground mass. But the dimensions of the base, h_p, depend on the applied moment, $M = F\ell$, where F is a force applied at the vehicle height. F can be due to either wind or to sudden braking of a stream of vehicles. Then, from equation (10.2.1) and equation (10.2.2) with $w = h >> t$,

$$\frac{M}{\sigma_m} = \frac{F\ell}{\sigma_m} = \frac{I}{c} = \frac{4}{3} h_p^2 t_p$$

Thus

$$h_p = \left(\frac{3F\ell}{4\sigma_m t_p} \right)^{1/2} \tag{10.9.3}$$

If F is due to wind,

$$F = \tfrac{1}{2}\rho_a V_w^2 (h + h_v)\ell_s$$

in which $(h + h_v)\ell_s$ is the cross-sectional area exposed to wind in one span length. If F is due to emergency deceleration of vehicles,

$$F = \frac{a_e W}{g}\frac{\ell_s}{\ell_h}$$

in which W is the weight of each vehicle, a_e is the emergency braking rate, and ℓ_h is the minimum headway of a stream of vehicles.

Based on the numerical values used in connection with equation (10.6.6), the maximum value of $\rho_a V_w^2(h + h_v)/2 \approx 1600$ N/m. Assuming $a_e/g = 0.5$, $W = 12600$ N (see figure 10-8), and $(\ell_h)_{\min} = 5$ m, $a_e W/g\ell_h = 630$ N/m. Assuming that for heavier vehicles, ℓ_h increases in proportion to W, the wind load dominates. Substituting the wind load into equation (10.9.3), and then equation (10.9.3) into equation (10.9.2), the cost of each post is

$$C_p = 2C_{p_{kg}}\rho V_w \left[\frac{1.5\rho_a \ell^3 t_p (h + h_v)}{\sigma_m} \right]^{1/2} \ell_s^{1/2} \qquad (10.9.4)$$

Now the installed cost of the whole guideway per unit length can be expressed as

$$C_T = C_g + \frac{C_p}{\ell_s}$$

Substituting equations (10.9.1) and (10.9.4),

$$C_T = C_{g0} + A_1 t(\ell_s) + A_2/\ell_s^{1/2} \qquad (10.9.5)$$

in which

$$A_1 = 2\rho C_{kg} h(1 + \alpha) \qquad (10.9.6)$$

and

$$A_2 = 2C_{pkg}\rho V_w \left[\frac{1.5\rho_a \ell^3 t_p (h + h_v)}{\sigma_m} \right]^{1/2} \qquad (10.9.7)$$

The optimum span length is found by differentiating equation (10.9.5) with respect to ℓ_s and setting the result equal to zero. Thus,

$$\frac{dt}{d\ell_s} = \frac{A_2/A_1}{2\ell_s^{3/2}} \tag{10.9.8}$$

gives the optimum value of ℓ_s.

As indicated above, it is necessary to take t at least as large as t_m which, from figure 10-3, corresponds to a specific value of ℓ_s, which we shall call ℓ_{s_m}. Then, from the form of equation (10.9.5), it is clear that the optimum value of ℓ_s must be at least as large as ℓ_{s_m}. Whether or not it is larger depends on whether or not the value of ℓ_s that satisfies equation (10.9.8) is larger or smaller than ℓ_{s_m}. If it is smaller, ℓ_{s_m} is the optimum value; if larger, the root of equation (10.9.8) is the optimum. Thus, assume $t(\ell_s)$ is the function given by equation (10.2.14) and express $t(\ell_s)$ in the form

$$t = \frac{ax^2}{1 - x^2} \tag{10.9.9}$$

where

$$a = \frac{3q_\ell}{8\rho g h} = \frac{3W}{8\rho g h \ell_v} \tag{10.9.10}$$

and

$$x = \ell_s/b \tag{10.9.11}$$

where

$$b = \left(\frac{2\sigma_m h}{\rho g}\right)^{1/2} \tag{10.9.12}$$

In the expression for a, we have assumed the live load per unit length is W/ℓ_v, where W is the vehicle weight and ℓ_v is its length. Now, from equation (10.9.9),

$$\frac{dt}{d\ell_s} = \frac{2ax}{b(1 - x^2)^2}$$

Substituting this expression into equation (10.9.8),

$$\frac{2ax}{b(1 - x^2)^2} = \frac{A_2/A_1}{2b^{3/2}x^{3/2}}$$

which may be written in the form

$$\mu = \frac{x^{5/2}}{(1 - x^2)^2} \tag{10.9.13}$$

where

$$\mu = \frac{A_2}{4ab^{1/2}A_1} = \frac{2(C_{pkg}/C_{kg})V_w\ell_v(\rho g)^{3/4}[1.5\rho_a\ell^3 t_p(h + h_v)]^{1/2}}{3(1 + \alpha)W(2h)^{1/4}\sigma_m^{3/4}} \tag{10.9.14}$$

Equation (10.9.13) is plotted in figure 10-9. Thus, after μ is found from equation (10.9.14), the optimum value of x is found as the corresponding value from figure 10-9. Then ℓ_s is found from equations (10.9.11) and (10.9.12). As indicated above, if this value of ℓ_s is less than ℓ_{s_m}, ℓ_{s_m} is the optimum value. Consider as an example a steel box beam for which $h = 1$ m, $\alpha = 1/3$. With $\rho = 7760$ kg/m^3, $\sigma_m = 140(10)^6$ N/m^2, $V_w = 27$ m/s, and $\rho_a = 1.293$ kg/m^3,

$$\mu = 15.9(C_{pkg}/C_{kg})\ell_v\ell^{3/2}t_p^{1/2}(h + h_v)^{1/2}/W$$

Assume $C\rho_{kg}/C_{kg} = 10$ to account for the cost of the entire post and its erection, and take W to be the smallest value given in figure 10-8, 12600 N. Then let $\ell_v = 2.6$ m, $h + h_v = 2.6$ m, and $\ell = 5$ m. Finally take $t_p = 0.02$ m to minimize possible damage in case of accidents, and we find $\mu = 0.084$. Then, from figure 10-9, $x = 0.335$. From equation (10.9.12), $b = 60.0$ m, Therefore, $\ell_s = 20.1$ m. But, if the wall thickness is limited to say 1 cm, figure 10-3 shows that the span is stress limited at $\ell_s = 32.8$ m. This is larger than the computed value of 20.1 m. Therefore, in this example, the guideway cost is minimum if ℓ_s is taken as 32.8 m. If it were possible to use plates less than one centimeter thick without a corresponding increase in fabrication cost per unit mass of material, the total cost could be decreased.

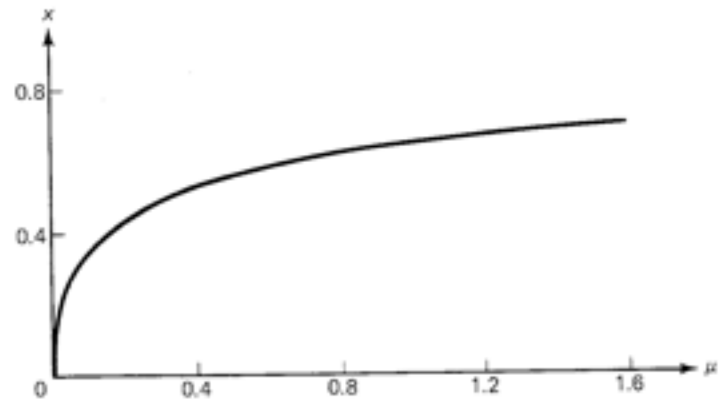

Figure 10-9. Span Length for Minimum Guideway Cost

10.10 Summary

The purpose of this chapter has been to examine the factors that have primary influence on the cost per unit length of elevated guideways, and to find optimum parameter choices where they exist that will minimize cost per unit length. Since the guideway is the single most expensive item in an exclusive-guideway transit system, the optimization of its parameters is of primary importance in minimizing the cost of the total system. While attention is devoted in this chapter exclusively to elevated guideways, it should be kept in mind that there may be circumstances in which at-grade guideways are satisfactory and would decrease system cost, and in which, in the case of small-vehicle systems, underground guideways may not be appreciably more expensive than elevated guideways [1].

The criteria for guideway design are primarily maximum stress and ride comfort; however, integration of the guideway into the system requires that consideration be given to the manner of switching and the manner of support of the vehicles by the guideway. To take advantage of the cost reduction possible with the use of small vehicles, it is necessary to consider switch mechanisms that do not require movement of a portion of the guideway; and, to provide adequate lateral stability and ride comfort, the dimensions of the guideway must not be too small. The results of this chapter indicate, however, that weight per unit length of the guideway increases too rapidly as parameters deviate from optimum to permit considerations of switching and lateral stability to dictate the basic guideway design. Rather, if weight and cost are to be minimized, optimum parameters must be chosen and then the switching and lateral stability requirements provided by clever design. Enough design and test experience has been accumulated to ascertain that these requirements are not incompatible. The visual appearance of the guideway is of course also an important factor in the design; however, this consideration is satisfied in a structurally optimum design because such a design has the smallest possible cross section.

It is not obvious a priori which stresses reach their design limits first in a given guideway design; therefore, all possibilities must be considered. The logical first step is to consider bending stresses produced by a static vertical load, assumed in the worst case to be due to vehicles at rest end-to-end on the guideway. Next, the dynamic loading due to motion of vehicles over the guideway is considered, first by studying the deflection, stress, and acceleration produced by a single vehicle of arbitrary speed and mass crossing a single, simply supported span, and second by studying the increased effects produced by cascades of vehicles crossing a span. The conclusion reached is that the static condition described above usually yields a higher stress than the dynamic stress produced with vehicles spaced farther apart, and therefore that, because the static yield point stress and fatigue limit stress are about the same in structural steel [7], the static condition of vehicles spaced end-to-end usually determines the design. Application of the ride comfort criterion then shows that at speeds higher than given by equation (10.5.4), the beam must be deeper than that required by the stress criterion.

Torsional stresses are next considered, and it is found that with a bending-optimum box beam, the torsional stresses due to wind and centrifugal forces are well below the design shear stress. Therefore only a small fraction of the depth of a bending-optimum box beam is needed to resist torsion, a conclusion that increases the flexibility of the choice of beam cross section. Slotted box beams are also studied in torsion, and it is found that for the same dimensions as a closed box beam, the shear stress is greater by a factor of about 75, thus requiring the walls of the beam to be substantially thicker. Finally, for thin-walled steel beams the possibility of plate buckling and plate vibration is examined. It is found that if the plate is thick enough so that its thinness will not cause fabrication problems (an intuitive judgement), buckling under the loads that can be produced by vehicles is not a problem, but that the plate can not practically be thick enough to avoid resonances in the audible range. Thus, it is necessary either to design the vehicle suspension system so that the plate vibratory modes will not be excited due to imperfections, or to apply an optimized vibration-damping layer to the surface of the beam. Such a treatment is described.

A number of numerical examples are worked out in this chapter to illustrate the method more specifically, and to give a feeling for the magnitudes of the various parameters. In all cases, the properties of ordinary structural steel are assumed. Perhaps the only other practical alternative is to use reinforced concrete; however, this is not done for three reasons: (1) the mathematics is much more straightforward for thin-walled beams (a good assumption for steel) than for thick-walled beams, which must be assumed with concrete; (2) there are more variables with concrete beams because of prestressing and placement of reinforcing bars; and (3) one example seems sufficient to illustrate the attainment of certain optimum conditions. Nonetheless, because of the low cost of reinforced concrete as compared to steel, its use should not be ignored and a similar solution should be carried through for concrete.

The results of chapter 10 will now be discussed in more detail: figure 10-2 gives the dimensionless material cross-sectional area $\mathscr{A}$ of a beam as a function of aspect ratio α for various values of a dimensionless loading factor $\mathscr{I}$. It is seen that for thin-walled beams the cross-sectional area of the beam for a given load is minimum if the aspect ratio if one-third. If the beam wall thickness is the same on all walls, this means that the width of the beam should be one-third the depth. As the wall thickness increases, the dimensionless loading factor $\mathscr{I}$ decreases and the optimum aspect ratio decreases. Thus, in all cases, a deep, narrow beam is indicated if the cost is to be minimized. By varying the parameter α in figure 10-2, one can see how the dimensionless area $\mathscr{A}$ increases away from the optimum condition. It can be expected that the cost per unit length of the beam increases with its cross-sectional area, that is, to the ordinate in figure 10-2; therefore this figure is basic to the determination of the relative cost of various beam designs. It is important to note that figure 10-2 is based on the fundamental bending stress formula, given by equation (10.2.1), and is independent of the manner or magnitude of loading.

Once the optimum aspect ratio of the beam is determined, its required thickness must be found as a function of load. If the wall is thin compared to the depth, the thickness is given by equation (10.2.14). For a given depth and span length, this equation shows that the required wall thickness is proportional to the live load per unit length. In the same circumstances, the cross-sectional area of the beam (and hence material cost per unit length) is proportional to the wall thickness. Therefore, if the beam is stress limited, which is the case for most urban applications, *the material cost per unit length of the guideway is proportional to the weight per unit length of the vehicles*. Figure 5-4 shows the mass per unit length of forty-seven different transit vehicles of various capacity, and together with the above conclusion, illustrates a major reason for interest in small vehicles in automated transit systems.

The required wall thickness is plotted from equation (10.2.14) in figure 10-3 for a live load of 447 kg/m (300 lb/ft). Since the wall thickness is proportional to live load, it can readily be determined for other loadings. Some of the wall thicknesses plotted may, as mentioned above, be too thin from the viewpoint of ease of fabrication. If, for example, the thinnest permissible sheet is 1 cm, a beam one-meter deep is stressed to the design limit only if the spans are longer than 32.8 m. For shorter spans, the beam could tolerate a higher unit loading without exceeding the design limit; however, the cost then increases because more posts than necessary are used. If the guideway is built up of two parallel box beams in order to support ordinary wheeled vehicles, it is shown that for the same span length and beam-wall thickness, the total cross-sectional area and hence cost per unit length increases by about 40 percent.

Once static loading is understood, it is necessary to determine if the motion of the vehicles will set up resonant conditions that will make dynamic loading more severe than the worst static loading case of vehicles parked end-to-end on the guideway. As mentioned above, this case is treated by first examining the motions produced by a single vehicle crossing the span. As indicated from figures 10-5 and 10-6, the greatest deflection at any speed is only about 70 percent greater than the static value produced with one vehicle parked at the center of the span. Thus, unless the vehicles are almost as long as the free spans, the maximum static load will be greater than the single-vehicle dynamic load. For dynamic loading with single or multiple vehicles, the resonant conditions are determined by the natural frequencies of vibration of the unloaded beam. The natural frequencies are proportional to $(I/A)^{1/2}$, where I is the moment of inertia of the cross section and A is the cross-sectional area. Thus, for a given A, and hence a given cost per unit length, resonant conditions cause the least difficulty if I is a maximum. Maximum I, in turn, is produced if the beam is as deep and narrow as possible, but one must take into consideration the need for lateral stiffness. It is shown that for the bending-optimum aspect ratio of one-third, the natural frequency in the horizontal plane is 43 percent of its value in the vertical plane, a value judged sufficiently high.

From analysis of cascades of vehicles crossing a span, based on an analysis performed at the Massachusetts Institute of Technology, it is found that the ratio of dynamic deflection to static deflection with the same vehicle loading is greater if the vehicles are farther apart and diminishes markedly as the headway between vehicles decreases (see figure 10-7). It is also found that the maximum amplitude of beam motion is amplified with successive vehicle crossings by a factor of about 1.9 in 15 to 20 crossings at which time a steady-state condition is reached. Based on the MIT results, if the minimum operating headway is 0.25, 0.5 times the span length, the worst-case static stress will be greater than the maximum dynamic stress (at any speed) if the span length is greater than 5.6, 13.8 times the vehicle length. But, if the maximum vehicle speed is kept about 20 percent below the primary resonant condition (usually attainable), the maximum dynamic deflection is only a small amount greater than the static deflection for the case of vehicles spaced a fourth of a span apart, and it can be concluded that the maximum static stress determines the design.

Figure 10-8 shows that the guideway mass per unit length, determined under dynamic conditions, increases roughly as the square root of the vehicle mass for a given flow in seats per hour. But, if the static stress determines the design, the guideway mass increases as the first power of the vehicle mass, and this is the usual condition. If the static condition and not the dynamic condition determines the design, a simple formula (equation (10.5.4)) is found that determines the maximum speed for which the ride comfort condition is met. In most cases it is found that this speed exceeds the maximum speed of interest in urban applications, and hence that the design is determined from static stress considerations.

Torsional stress considerations indicate, as mentioned above, that a full box beam is not needed to resist the torsional loads, thus permitting some freedom in the design. If the beam is slotted, however, torsional loads are of prime importance. In this case, because of the added vertical height of standing-passenger vehicles and the accompanying increase in wind load, the use of standing-passenger vehicles adds about 40 percent to the mass per unit length of the beam required to resist the wind load.

Finally, a formula is found (equations (10.9.13, 10.9.14)) for the span length that minimizes the total cost of beams plus support posts.

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