3 Transitions from Straight to Curved Guideways

Chapter 2 deals with longitudinal performance relationships which can be applied on both straight and curved sections of guideway. To understand the layout of the specific network configurations discussed in chapter 4, we also need to consider what may be called "lateral performance relationships." These deal with limitations on lateral curvature and rate of change of curvature of guideways due to comfort limitations on lateral acceleration and jerk.

If the stiffness of lateral support between the vehicle and guideway is high, the lateral jerk limitation results in a requirement for spiral transition sections from straight to curved sections of guideway. Spiral transitions will be treated first. Among these there are two types of practical importance: one in which the velocity of the vehicle is constant, and the other in which the vehicle is subject to constant deceleration or acceleration.

If it is practical to reduce the stiffness of the lateral vehicle support device, abrupt changes in guideway curvature can be tolerated under certain conditions. Since allowing these abrupt changes may reduce the cost of manufacture of the guideway, the conditions under which they can be tolerated are derived.

Finally, the minimum radius of curvature of a guideway can be reduced if the curve is superelevated. Reducing the minimum radius of curvature permits greater freedom of design of networks in street systems, reduces the possibility that buildings will have to be removed at curves, and reduces the length and hence, the cost of curves. For these reasons, formulas for design of superelevated curves are derived.

3.1 The Differential Equation for the Transition Curve

Consider the curve shown in figure 3-1, which passes through the origin of the $x - y$ coordinates with zero slope and zero curvature. The arc length from the origin to an arbitrary point P is s, the angle between the velocity vector $\mathbf{V}$ and the x-axis at P is θ, the tangential unit vector in the direction of $\mathbf{V}$ is $\hat{t}$, and the normal unit vector is $\hat{n}$. As point P moves to the right at velocity $\mathbf{V}$, the unit vectors rotate according to the relationships

$$\begin{aligned} d\hat{t} &= \hat{n}d\theta \\ d\hat{n} &= -\hat{t}d\theta \end{aligned} \tag{3.1.1}$$

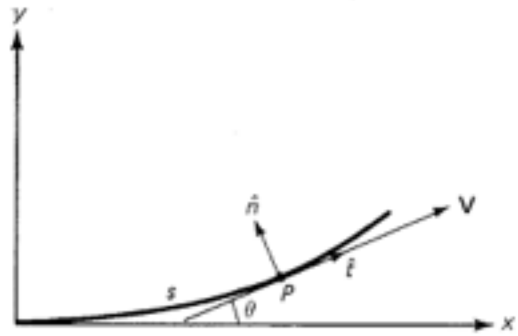


Figure 3-1. Notation in a Transition Curve

Let V be the magnitude of $\mathbf{V}$, $\mathbf{a}$ be the acceleration of point P, and $\mathbf{J}$ be its jerk. Then

$$\mathbf{V} = V\hat{t} \tag{3.1.2}$$

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \frac{dV}{dt}\,\hat{t} + V\,\frac{d\theta}{dt}\,\hat{n} \tag{3.1.3}$$

$$\mathbf{J} = \frac{d\mathbf{a}}{dt} = \frac{d^2V}{dt^2}\,\hat{t} + 2\,\frac{dV}{dt}\,\frac{d\theta}{dt}\,\hat{n}$$

$$+ V\frac{d^2\theta}{dt^2}\hat{n} - V\left(\frac{d\theta}{dt}\right)^2\hat{t} \tag{3.1.4}$$

From equation (3.1.4), the tangential jerk is

$$J_t = \frac{d^2V}{dt^2} - V\left(\frac{d\theta}{dt}\right)^2 \tag{3.1.5}$$

and the normal jerk is

$$J_n = V\,\frac{d^2\theta}{dt^2} + 2\,\frac{dV}{dt}\,\frac{d\theta}{dt} \tag{3.1.6}$$

We wish to determine the shape of the guideway (the curve of figure 3-1), therefore, we make the transformation

$$\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds} = V\frac{d}{ds} \tag{3.1.7}$$

hence

$$\frac{d\theta}{dt} = V\frac{d\theta}{ds} \tag{3.1.8}$$

and

$$\frac{d^2\theta}{dt^2} = V^2\frac{d^2\theta}{ds^2} + \frac{dV}{dt}\frac{d\theta}{ds} \tag{3.1.9}$$

and equations (3.1.5) and (3.1.6) can be written

$$J_t = \frac{d^2V}{dt^2} - V^3\left(\frac{d\theta}{ds}\right)^2 \tag{3.1.10}$$

$$J_n = V^3\frac{d^2\theta}{ds^2} + 3V\frac{dV}{dt}\frac{d\theta}{ds} \tag{3.1.11}$$

Equation (3.1.11) with J_n constant defines the transition curve. Once it is found, J_t can be found from equation (3.1.10) to determine if it exceeds the comfort criterion.

In practical cases we will consider $d^2V/dt^2 = 0$; therefore, equation (3.1.10) becomes

$$J_t = -\frac{1}{V}\left(V^2\frac{d\theta}{ds}\right)^2 \tag{3.1.12}$$

But from equation (3.1.3) and (3.1.8), the normal acceleration is

$$a_n = V^2\frac{d\theta}{ds} = \frac{V^2}{R} \tag{3.1.13}$$

in which $d\theta/ds$ is the curvature and R is the radius of curvature. Thus, equation (3.1.12) becomes

$$J_t = - \frac{a_n^2}{V} \tag{3.1.14}$$

If the limit value of a_n and the minimum value of V are substituted into equation (3.1.14), and the result is below the jerk limit, the acceleration limit determines the length of the spiral transition; otherwise, the limit is determined by jerk. Since the acceleration and jerk limits are approximately equal in units of seconds, the acceleration limit governs if a_n is less than V_{min}, a condition which is usually satisfied.

3.2 The Constant Speed Spiral

If V is constant, equation (3.1.11) can be written

$$\frac{d^2\theta}{ds^2} = \frac{J_n}{V^3} \tag{3.2.1}$$

At $s = 0$, $\theta = d\theta/ds = 0$; therefore, the curvature is

$$\frac{d\theta}{ds} = \frac{J_n s}{V^3} \tag{3.2.2}$$

and, from equation (3.1.13), the normal acceleration is

$$a_n = \frac{J_n s}{V} \tag{3.2.3}$$

Integrating equation (3.2.2), we have

$$\theta = \frac{J_n s^2}{2V^3} \tag{3.2.4}$$

If equation (3.2.3) is solved for s and substituted into equation (3.2.4),

$$\theta = \frac{a_n^2}{2J_n V} \tag{3.2.5}$$

With the limit values of a_n and J_n substituted into equations (3.2.5) and

(3.2.3), we obtain the maximum values of θ and s, respectively, along the constant velocity spiral. With $a_n = J_n$ in units of seconds,

$$\theta_{max} = \frac{a_n}{2V} \tag{3.2.6}$$

and

$$s_{max} = V \tag{3.2.7}$$

The equation of the constant velocity spiral in rectangular coordinates (x, y) is found from the differential relationships

$$\left.\begin{aligned} dx &= ds \cos \theta \\ dy &= ds \sin \theta \end{aligned}\right\} \tag{3.2.8}$$

in which all terms are defined in figure 3-1.

The equation of the spiral transition section is therefore given parametrically by the equations

$$\left.\begin{aligned} x &= \int_0^s \cos \theta(s)\, ds \\ y &= \int_0^s \sin \theta(s)\, ds \end{aligned}\right\} \tag{3.2.9}$$

in which $\theta(s)$ is given by equation (3.2.4).

The angle $\theta(s)$ is limited to the value given by equation (3.2.6). In an extreme case, we can assume $a_n = 2.5$ m/s^2 and $V = 5$ m/s^3. In this case, $\theta_{max} = 0.25$ radian. In most cases, θ_{max} is much smaller; therefore, use of only the first term in the Taylor series expansions of the sine and cosine is sufficient. At θ_{max}, the second terms in the Taylor series expansions

$$\cos \theta = 1 - \frac{\theta^2}{2} + \ldots$$

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \ldots$$

produce an error less than $\theta_{max}^2/2 = 1/32$ compared to unity. Therefore, substituting $\cos \theta = 1$ and $\sin \theta = \theta$ into equations (3.2.9), and then equation (3.2.4), we have

$$\left.\begin{aligned} x &= s \\ y &= \frac{J_n s^3}{6V^3} \end{aligned}\right\} \tag{3.2.10}$$

The equation of the spiral is, therefore

$$y = \frac{J_n x^3}{6V^3} \tag{3.2.11}$$

If we define the dimensionless variables

$$\begin{aligned} \bar{s} &= \frac{s}{\sqrt{2V^3/J_n}} \\ \bar{x} &= \frac{x}{\sqrt{2V^3/J_n}} \\ \bar{y} &= \frac{y}{\sqrt{2V^3/J_n}} \end{aligned} \tag{3.2.12}$$

equation (3.2.4) becomes

$$\theta = \bar{s}^2 \tag{3.2.13}$$

and equation (3.2.11) becomes

$$\bar{y} = \frac{\bar{x}^3}{3} \tag{3.2.14}$$

Since these equations contain no parameter, we see that the family of constant velocity spirals scale in proportion to the parameters $\sqrt{2V^3/J_n}$, that is, in proportion to $V^{3/2}$. From equation (3.2.7) we note, however, that the maximum length of the constant speed spiral is proportional to V.

3.3 A Right-Angle Curve at Constant Speed

In this section, the theory of section 3.2 is applied to the specification of a right-angle curve in which the vehicles are to maintain constant line speed V. A constant speed spiral forms the transition from a straight guideway to a guideway of constant radius of curvature R, which, from equation (3.1.13), is

$$R = \frac{V^2}{a_n} \tag{3.3.1}$$

in which, a_n is specified from comfort conditions. A second spiral, which is the mirror image of the first rotated 90 degrees counterclockwise forms the transition from the constant curvature section back to a straight section. The problem of this section is to determine the coordinates required to lay out the entire curve, and the length of the curved sections.

Let the origin of the $(x - y)$ coordinates be at the point of transition from the straight section to the spiral transition section, with the velocity vector at the origin pointed in the $+x$ direction. Then the equation of the first transition spiral is, without transformation, equation (3.2.11). Call the end point of this transition section (x_1, y_1). Then the coordinates x_1 and y_1 are found by substituting $s_{\max}$ from the equation (3.2.7) into equation (3.2.10).

Thus

$$\left.\begin{aligned} x_1 &= V \\ y_1 &= \frac{J_n}{6} \end{aligned}\right\} \tag{3.3.2}$$

The length of the first transition section is V in units of seconds, and the guideway at (x_1, y_1) makes an angle θ, with the x-axis, where from equation (3.2.6)

$$\theta_1 = \frac{a_n}{2V} \tag{3.3.3}$$

Let (x_2, y_2) be the coordinates of the center of curvature of the section of constant curvature. Then, from a simple geometric construction,

$$x_2 = x_1 - R \sin \theta_1$$

$$y_2 = y_1 + R \cos \theta_1$$

Since θ_1 is a small angle, let $\sin \theta_1 = \theta_1$ and, $\cos \theta_1 = 1$. Then substitute from equations (3.3.1), (3.3.2) and (3.3.3) to obtain

$$\left.\begin{aligned} x_2 &= \frac{V}{2} \\ y_2 &= \frac{J_n}{6} + \frac{V^2}{a_n} \end{aligned}\right\} \tag{3.3.4}$$

Let (x_3, y_3) be the coordinates of the center point of the section of constant curvature. This point is important because it determines the clearance required for the curve. From a geometric construction,

$$x_3 = x_2 + R/\sqrt{2}$$

$$y_3 = y_2 - R/\sqrt{2}.$$

Substituting equations (3.3.1) and (3.3.4),

$$\left.\begin{aligned} x_3 &= \frac{V}{2} + 0.707\,\frac{V^2}{a_n} \\ y_3 &= \frac{J_n}{6} + 0.293\,\frac{V^2}{a_n} \end{aligned}\right\} \qquad (3.3.5)$$

Let (x_4, y_4) be the coordinates of the end point of the section of constant curvature. Then

$$x_4 = x_2 + R\cos\theta_1$$

$$y_4 = y_2 - R\sin\theta_1$$

Making the small angle assumption and substituting from equations (3.3.1), (3.3.3), and (3.3.4),

$$\left.\begin{aligned} x_4 &= \frac{V}{2} + \frac{V^2}{a_n} \\ y_4 &= \frac{J_n}{6} + \frac{V^2}{a_n} - \frac{V}{2} \end{aligned}\right\} \qquad (3.3.6)$$

Finally, let (x_5, y_5) be the end point of the spiral transition from curved back to straight guideway. Then,

$$x_5 = x_4 + y_1$$

$$y_5 = y_4 + x_1$$

Substituting from equations (3.3.2) and (3.3.6),

$$x_5 = y_5 = \frac{J_n}{6} + \frac{V}{2} + \frac{V^2}{a_n} \qquad (3.3.7)$$

The length of the section of constant curvature is $R(\pi/2 - 2\theta_1)$, therefore, using equations (3.3.1), (3.3.3), and (3.2.7), the total length of curved guideway is

$$\text{Curved Guideway Length} = R\,(\pi/2 - 2\theta_1) + 2V$$

$$= \frac{\pi}{2}\frac{V^2}{a_n} + V \qquad (3.3.8)$$

Thus, the addition of a spiral transition adds a length V (in units of seconds) to the total length of curved guideway.

3.4 Transition to an Off-Line Station at Constant Speed

In this section, the theory of section 3.2 is applied to the design of a constant speed transition from a mainline guideway onto a parallel guideway separated by a distance H from the mainline. The transition, shown in figure 3-2, is made up of four constant speed spirals of the type given by equation (3.2.11), connected so that the slope and curvature are everywhere continuous. We let the total length of the transition section in the direction of flow be denoted by L.

The section of the transition shown in figure 3-2, between $x = 0$, and $x = L/4$, is computed from equation (3.2.11) without transformation. The curvature is a maximum at point $x = L/4$ and vanishes at points $x = 0, L/2$. Therefore, the transition section from $x = L/4$ to $x = L/2$ is a mirror image of the first section about the perpendicular bisector of the line connecting the origin with the point $x = L/2$, $y = H/2$. The section from $x = L/2$ to $x = L$ is obtained by rotating the first half of the transition 180 degrees in the plane of the paper about the midpoint.

$$\frac{H}{L} = \left.\frac{dy}{dx}\right|_{x=\frac{L}{4}} = \frac{J_n x^2}{2V^2} = \frac{J_n L^2}{32 V^2} \; ; \quad L_{min} = 4V\left(\frac{H}{2J_n}\right)^{1/3}$$

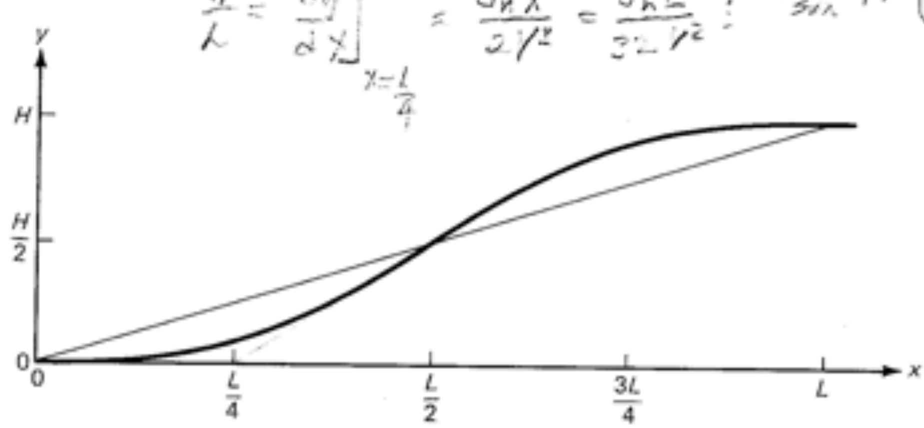


Figure 3-2. A Spiral Transition to a Parallel Line at Constant Speed

In the derivation of the relationship between L and H, it is convenient to use the dimensionless notation defined by equations (3.2.12). Thus, let

$$\left.\begin{aligned} \bar{H} &= \frac{H}{\sqrt{2V^3/J_n}} \\ \text{and} \qquad \bar{L} &= \frac{L}{\sqrt{2V^3/J_n}} \end{aligned}\right\} \tag{3.4.1}$$

and consider the dimensionless form of the equation for the spiral, equation (3.2.14). From figure 3-2, and equation (3.2.14), we see that at $\bar{x} = \bar{L}/4$

$$\frac{d\bar{y}}{d\bar{x}} = \bar{x}^2 = \frac{\bar{H}}{\bar{L}}$$

Substituting $\bar{x} = \bar{L}/4$, we have

$$\bar{H} = \frac{\bar{L}^3}{16} \tag{3.4.2}$$

Since equation (3.4.2) and equation (3.2.14) contain only the dimensionless values and no parameters, the transition spiral scales in proportion to the parameter $\sqrt{2V^3/J_n}$, that is, in proportion to $V^{3/2}$. Substituting equations (3.4.1) into equation (3.4.2), we find that

$$L = V\left(\frac{32H}{J_n}\right)^{1/3} \tag{3.4.3}$$

Thus, for a given value of H, L increases in proportion to V.

The maximum magnitude of the normal acceleration a_n occurs at $x = L/4$ and at $x = 3L/4$. Therefore, in equation (3.2.3), substitute $s = x = L/4$, and then equation (3.4.3). We obtain

$$a_n = \frac{J_n L}{4V} = \left(\frac{J_n^2 H}{2}\right)^{1/3} \tag{3.4.4}$$

Hence, for given maximum values of J_n and a_n, the maximum permissible value of H is

$$H_{\max} = \frac{2a_n^3}{J_n^2} = 2a_n \tag{3.4.5}$$

in seconds units if $(a_n)_{\max} = J_n$. If a lateral displacement larger than $H_{\max}$ is required, a straight section must be inserted at $x = L/2$ in figure 3-2. From symmetry, the slope of the straight section is $2H/L$. If $H_{\max}$ is substituted into equation (3.4.3) and $a_n = J_n$,

$$L_{\max} = 4V \tag{3.4.6}$$

The minimum radius of curvature is found by substituting equation (3.4.4) into equation (3.1.13). Thus

$$R_{\min} = V^2 \left(\frac{2}{J_n^2 H} \right)^{1/3} \tag{3.4.7}$$

3.5 The Constant Deceleration Spiral

This case is defined by the equation

$$\frac{dV}{dt} = -a \tag{3.5.1}$$

Substituting equation (3.5.1) into equation (3.1.11) gives, J_n constant, the equation of the constant deceleration spiral. This equation can be integrated if we note that, by substituting equation (3.1.7) into equation (3.5.1),

$$ds = \frac{-V\,dV}{a} \tag{3.5.2}$$

from which

$$\frac{d\theta}{ds} = -\frac{a}{V}\frac{d\theta}{dV} \tag{3.5.3}$$

and

$$\frac{d^2\theta}{ds^2} = \frac{a^2}{V}\frac{d}{dV}\left(\frac{d\theta}{V\,dV}\right) \tag{3.5.4}$$

Thus, with V as the independent variable, equation (3.1.11) becomes

$$a^2V^2\frac{d}{dV}\left(\frac{d\theta}{V\,dV}\right) + 3a^2\frac{d\theta}{dV} = J_n$$

If we multiply both sides of this equation by V, the left side becomes a perfect differential:

$$\frac{d}{dV}\left(V^2 \frac{d\theta}{dV}\right) = \frac{J_n V}{a^2} \tag{3.5.5}$$

The initial conditions at $s = o$ are $V = V_0$and $\theta = d\theta/ds = 0$; thus, from equation (3.5.3), $d\theta/dV = 0$. Therefore, the integral of equation (3.5.5) can be written

$$\frac{d\theta}{dV} = -\frac{J_n}{2a^2}\left(\frac{V_0^2}{V^2} - 1\right) \tag{3.5.6}$$

Integrating again, θ can be written in the form

$$\theta = \frac{J_n V_0}{a^2} \frac{(1 - V/V_0)^2}{2V/V_0} \tag{3.5.7}$$

By substituting equation (3.5.6) into equation (3.5.3), we obtain the curvature of the decelerating spiral.

$$\frac{1}{R} = \frac{d\theta}{ds} = \frac{J_n}{2aV_0} \frac{V_0}{V}\left(\frac{V_0^2}{V^2} - 1\right) \tag{3.5.8}$$

in which R is the radius of curvature. Using equation (3.1.13), the normal acceleration is

$$a_n = \frac{V^2}{R} = \frac{J_n}{2a} V\left(\frac{V_0^2}{V^2} - 1\right) \tag{3.5.9}$$

With the limit value of a_n substituted, the minimum value of V is the positive root of equation (3.5.9) solved for V:

$$V_{\min} = \frac{aa_n}{J_n}\left[\sqrt{1 + \left(\frac{V_0 J_n}{aa_n}\right)^2} - 1\right] \tag{3.5.10}$$

Substitution of equation (3.5.10) into equations (3.5.7) and (3.5.8) gives the maximum value of θ and the minimum value of R, respectively.

As an example, let $J_n = a_n = a = 2.5$ m/s², and $V_0 = 10$ m/s. Then, from equation (3.5.10), $V_{min} = 7.81$ m/s. Substituting V_{min} into equations (3.5.7) and (3.5.8),

$$\theta_{max} = 7.05° \text{ and } R_{min} = 24.4 \text{ m}$$

In practical cases, speed is decreased to obtain the minimum radius of curvature, and hence the smallest requirement for clearance. In this case, both a_n and R_{min} are specified, and V_m is found from the equation

$$V_{min} = \sqrt{a_n R_{min}} \tag{3.5.11}$$

Then, from equation (3.5.9), the velocity at the beginning of the spiral transition should be

$$V_0 = V_{min}\left(1 + \frac{2aa_n}{J_n V_{min}}\right)^{1/2} \tag{3.5.12}$$

that is, the line speed should be slowed to V_0 before entering the spiral.

The length of the decelerating spiral is found by integrating equation (3.5.2) with the initial condition $s = 0$ when $V = V_0$. The result may be written

$$s = \frac{V_0^2}{2a}\left(1 - \frac{V^2}{V_0^2}\right) \tag{3.5.13}$$

Then, in the above example, the maximum length of the spiral is obtained by substituting into equation (3.5.13) the values $V_0 = 10$ m/s, $V = 7.81$ m/s, and $a = 2.5$ m/s². Then $s_{max} = 7.8$ m.

The equation of the spiral is found by substituting equation (3.5.7) into equation (3.2.8) in which we substitute equation (3.5.2). The resulting equation can be integrated in dimensionless form if we define the following dimensionless variables:

$$\beta = \frac{J_n V_0}{2a^2} \tag{3.5.14}$$

$$\xi = \frac{V}{V_0} \tag{3.5.15}$$

$$\left.\begin{aligned} X &= ax/V_0^2 \\ Y &= ay/V_0^2 \end{aligned}\right\} \qquad (3.5.16)$$

Thus,

$$\left.\begin{aligned} X &= \int_{V/V_0}^{1} \cos\left[\frac{\beta\,(1-\xi)^2}{\xi}\right]\xi\,d\xi \\ Y &= \int_{V/V_0}^{1} \sin\left[\frac{\beta\,(1-\xi)^2}{\xi}\right]\xi\,d\xi \end{aligned}\right\} \qquad (3.5.17)$$

If it can be assumed that $\theta^2/2$ is much less than 1, these equations become:

$$\left.\begin{aligned} X &= \int_{V/V_0}^{1} \xi\,d\xi = ½\left(1 - \frac{V^2}{V_0^2}\right) \\ Y &= \beta\int_{V/V_0}^{1} (1-\xi)^2 d\xi = \frac{\beta}{3}\left(1 - \frac{V}{V_0}\right)^3 \end{aligned}\right\} \qquad (3.5.18)$$

Solving the first of these equations for V/V_0 and substituting into the second, we have the equation of the decelerating spiral for small angles:

$$Y = \frac{\beta}{3}\left[1 - (1 - 2X)^{1/2}\right]^3 \qquad (3.5.19)$$

The specification of a right angle curve with deceleration is found by following the procedure of section 3.3, in which the coordinates of the endpoint of the spiral are found from equations (3.5.18) and the endpoint angle from equation (3.5.7), both for the appropriate value of V_{min}/V_0.

3.6 The Lateral Response of a Vehicle due to a Sudden Change in the Curvature of the Path

In some cases, spiral guideways have been found to be more expensive to manufacture than guideways of constant curvature. Therefore, it is useful

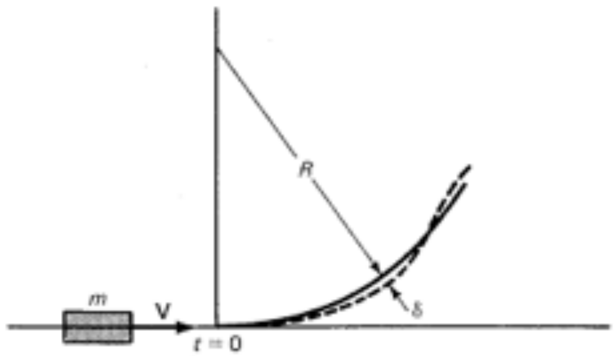


Figure 3-3. A Sudden Transition from a Straight to a Curved Guideway

to know the conditions under which spiral sections can be approximated by sections of constant curvature. The problem reduces to the determination of the lateral response of a vehicle due to a sudden change in the curvature of the path.

Consider a vehicle of mass m moving to the right in figure 3-3 with speed V. The vehicle has a lateral suspension system with spring constant k, damping coefficient ζ, and maximum permissible lateral deflection δ_m. At the point $t = 0$, the guideway curvature suddenly changes from zero to $1/R$. For t greater than 0, the acceleration of the vehicle (and passengers) in the direction normal to the curved path is $\ddot{\delta} - V^2/R$ in which δ is the deflection of the lateral suspension system, positive if away from the center of curvature as indicated in figure 3-3. The lateral equation of motion can be written in the form

$$\ddot{\delta} + 2\zeta\omega\dot{\delta} + \omega^2\delta = a_c \qquad (3.6.1)$$

in which

$$\omega = \sqrt{\frac{k}{m}} \qquad (3.6.2)$$

and

$$a_c = \frac{V^2}{R} \qquad (3.6.3)$$

Also, let

$$\omega' = \omega\sqrt{1 - \zeta^2} \qquad (3.6.4)$$

Then, subject to the initial conditions $\delta(0) = \dot{\delta}(0) = 0$, the solution to equation (3.6.1) is

$$\delta = \frac{a_c}{\omega^2}\left[1 - e^{-\omega\zeta t}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega' t + \cos\omega' t\right)\right] \tag{3.6.5}$$

Differentiating,

$$\dot{\delta} = \frac{a_c}{\omega\sqrt{1-\zeta^2}}\,e^{-\omega\zeta t}\sin\omega' t \tag{3.6.6}$$

$$\ddot{\delta} = a_c e^{-\omega\zeta t}\left(\cos\omega' t - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega' t\right) \tag{3.6.7}$$

$$\dddot{\delta} = -\,a_c\omega e^{-\omega\zeta t}\left[\frac{(1-2\zeta^2)}{\sqrt{1-\zeta^2}}\sin\omega' t + 2\zeta\cos\omega' t\right] \tag{3.6.8}$$

$$\ddddot{\delta} = -\,a_c\omega^2 e^{-\omega\zeta t}\left[(1-4\zeta^2)\cos\omega' t - \frac{\zeta(3-4\zeta^2)}{\sqrt{1-\zeta^2}}\sin\omega' t\right] \tag{3.6.9}$$

Note that the lateral acceleration of the vehicle is $\ddot{\delta} - a_c$, which is zero at $t = 0$.

The maximum value of δ occurs at the first zero of $\dot{\delta}$ for t greater than 0, which, from equation (3.6.6), occurs at $\omega' t = \pi$. Substituting this value in equation (3.6.5),

$$\delta_m = a_c/\omega^2\,(1 + e^{-\pi\zeta/\sqrt{1-\zeta^2}}) \tag{3.6.10}$$

With δ_m given by design, k should be chosen for a given m (see equation (3.6.2) so that

$$\omega^2 = \frac{a_c}{\delta_m}\left(1 + e^{-\pi\zeta/\sqrt{1-\zeta^2}}\right) \tag{3.6.11}$$

in which ζ is yet to be determined. By setting $\dddot{\delta} = 0$, solving for $\omega' t$, and substitution into equation (3.6.7), it can be shown that the maximum lateral acceleration of the passengers, $|\ddot{\delta} - a_c|$, occurs at $t = \infty$.[a] Thus, to satisfy the comfort criterion, we need to compute the maximum value of jerk, $\dddot{\delta}$. The maximum value of the function $\dddot{\delta}(\omega' t)$ corresponds to the first zero of $\ddddot{\delta}$.

[a] True if $\zeta > 1/\sqrt{2}$. If $\zeta < 1/\sqrt{2}$, $a_{max} > a_c$. If $\zeta = 1/3$, $a_{max} = 1.419\,a_c$.

From equations (3.6.8) and (3.6.9), we see that if $\zeta = 0$, $\dddot{\delta}(0) = 0$ and the first maximum in $\dddot{\delta}$ occurs at $\omega' t = \pi/2$. As the damping ratio increases, the first zero of $\ddddot{\delta}(\omega' t)$ moves to earlier values of $\omega' t$ until at $\zeta = 0.5$ the first zero of $\ddddot{\delta}(\omega' t)$ occurs at $\omega' t = 0$. For ζ slightly larger than 0.5, both terms in equations (3.6.9) are negative from $\omega' t = 0$ to a value slightly less than $\omega' t = \pi$. But, because of the exponential decay term, the value $\dddot{\delta}(0)$ is greater than at the first zero of $\ddddot{\delta}$ for $\omega' t$ greater than 0. At $\zeta = \sqrt{3/4}$, the first zero of $\ddddot{\delta}$ has moved back to $\omega' t = \pi/2$, but again $\dddot{\delta}(0)$ greater than $\dddot{\delta}(\pi/2)$. Thus, for $0 < \zeta <$ ½, the maximum value of $\dddot{\delta}(\omega' t)$ is found by setting $\ddddot{\delta}(\omega' t) = 0$. From equation (3.6.9), we then find

$$\tan \omega' t = \frac{\sqrt{1 - \zeta^2}}{\zeta} \left(\frac{1 - 4\zeta^2}{3 - 4\zeta^2} \right) \tag{3.6.12}$$

If we use the trigonometric identity $\cos \theta = (1 + \tan^2 \theta)^{-1/2}$ and substitute equation (3.6.12) into equation (3.6.8), the bracketed term reduces to unity, and the maximum jerk becomes

$$J_m = a_c \omega \exp \left\{ \frac{-\zeta}{\sqrt{1 - \zeta^2}} \tan^{-1} \left[\frac{\sqrt{1 - \zeta^2}}{\zeta} \left(\frac{1 - 4\zeta^2}{3 - 4\zeta^2} \right) \right] \right\} \tag{3.6.13}$$

If $\zeta >$ ½, the maximum jerk is

$$\dddot{\delta}(0) = J_m = 2 a_c \omega \zeta \tag{3.6.14}$$

In general, let

$$J_m = a_c \omega \, F(\zeta) \tag{3.6.15}$$

in which the meaning of $F(\delta)$ is found from equation (3.6.13) or (3.6.14). Then square equation (3.6.15) and substitute for ω^2 from equation (3.6.11). The results may be written

$$J_m^2 = \frac{a_c^3}{\delta_m} \left(1 + e^{-\pi\zeta/\sqrt{1-\zeta^2}} \right) F^2(\zeta) \tag{3.6.16}$$

We wish to know how small the radius of curvature, R, can be before J_m

reaches the comfort limit. Therefore, solve equations (3.6.3) for R and substitute for a_c from equation (3.6.16). The result may be written

$$R = \frac{V^2}{J_m^{2/3}\delta_m^{1/3}} \quad G(\zeta) \tag{3.6.17}$$

in which

$$G(\zeta) = \left[1 + \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)\right]^{1/3} \times$$

$$\exp\left\{ \frac{-2\zeta}{3\sqrt{1-\zeta^2}} \tan^{-1}\left[\frac{\sqrt{1-\zeta^2}}{\zeta}\left(\frac{1-4\zeta^2}{3-4\zeta^2}\right)\right]\right\} \qquad (0 \leq \zeta \leq 1/2)$$

$$= \left[1 + \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)\right]^{1/3} (2\zeta)^{2/3} \qquad (1/2 \leq \zeta \leq 1) \tag{3.6.18}$$

The choice of ζ depends on the degree of damping desired, which can be measured by the ratio of the second extremum in the function $|\delta(\omega' t) - \delta(\infty)|$ to the first. Thus,

$$\frac{\delta_{m_2}}{\delta_{m_1}} = \left| \frac{\delta(2\pi) - \delta(\infty)}{\delta(\pi) - \delta(\infty)} \right| = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) \tag{3.6.19}$$

The function $\delta_{m_2}/\delta_{m_1}$ and $G(\zeta)$ are plotted in figure 3-4.

Figure 3-4 together with equation (3.6.17) show that the radius of curvature that can be negotiated for a given comfort criterion, given by J_m, is minimized if $\zeta = 1/3$.[6] At this value, the ratio $\delta_{m_2}/\delta_{m_1} = 0.329$, which would appear to be a satisfactory degree of damping. The minimum value of $G(\zeta)$ is $G(1/3) = 0.966$. Therefore, from equation (3.6.17),

$$R_{\min} = 0.966 \frac{V^2}{J_m^{2/3}\delta_m^{1/3}} \tag{3.6.20}$$

[6] $\zeta = .3330$

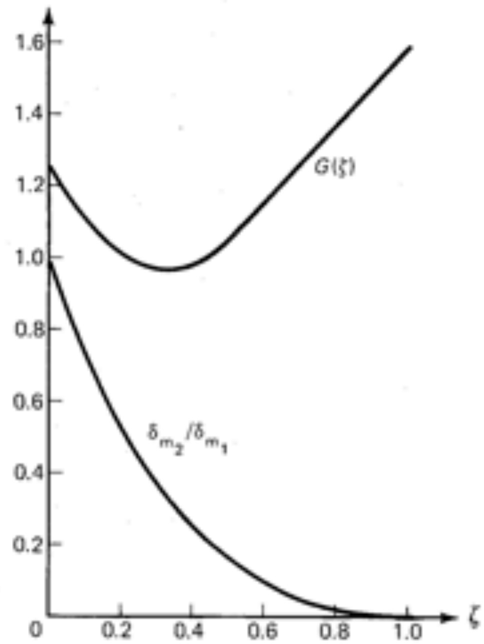


Figure 3-4. Lateral Damping Functions

As an example, let $V = 10$ m/s, $J_m = 2.5$ m/s^3, and $\delta_m = 0.05$ m. Then $R_{min} =$ 142 m. Substituting into equation (3.6.3), $a_c = 0.70$ m/s^2 which is less than the acceleration limit $(a_m)_{max} = J_m$. Therefore, the curve is determined by the jerk limit and R is given by equation (3.6.20). In general, R is determined by the jerk limit, not the acceleration limit if

$$\frac{J_m^{2/3}\delta_m^{1/3}}{0.966} < (a_c)_{lim} = a_{max}/1.419$$

But $J_m = (a_c)_{lim}$ in units of seconds. Therefore

$$\delta_m^{1/3} < 0.966\, J_m^{1/3}$$

or

$$\delta_m < 0.901\, J_m$$

if the limit is determined by jerk. Thus, if δ_m is less than approximately ~~one~~ 0.8

meter, certainly always true, the radius of curvature is limited by equation (3.6.20).

An additional interpretation of equation (3.6.20) is found by solving it for J_m:

$$J_m = 0.949 \quad \frac{V^3}{R^{3/2}\delta_m^{1/2}} \tag{3.6.21}$$

For a given lateral suspension system defined by δ_m and ζ, and given abrupt changes in curvature characterized by R, equation (3.6.21) shows that the uncomfortableness of the ride, characterized by J_m, worsens as the cube of the velocity.

Suppose a vehicle is on a path of curvature $1/R_1$, and suddenly enters a path of curvature $1/R_2$, in which $R_2 < R_1$. Then, if R is given by equation (3.6.20), the minimum value of R_2 that will meet the comfort criterion is found from the equation

$$\frac{1}{R_2} - \frac{1}{R_1} = \frac{1}{R}$$

from which

$$R_2 = \frac{R_1 R}{R_1 + R} \tag{3.6.22}$$

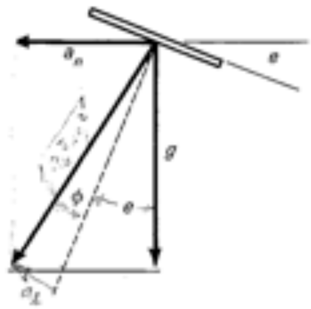
If R_2 is substituted into equation (3.6.3) and the computed value of a_c is less than the limit value, equation (3.6.22) determines the curve. Suppose we wish to design a right-angle turn in the guideway in circular arc segments so that the criterion on maximum lateral jerk is always satisfied. Then, from equation (3.6.22), the radius of curvature of successive segments are $R_2 = R$, $R/2$, $R/3$, $R/4$, and so on. In practice, however, it is unlikely that more than two different curvatures will be used.

3.7 Superelevation

The minimum radius in a turn at a given speed can be reduced by means of superelevation. Consider a superelevated, curved guideway, a cross section of which is shown in figure 3-5 at a point at which the speed is V and the superelevation angle is e. The resultant of the vectors representing the centrifugal force $\mathbf{a}_n$, and the gravity force $\mathbf{g}$, makes an angle φ with the normal to the floor of the vehicle. ~~It is the angle φ~~ that is specified to meet comfort criteria. From figure 3-5 we have

$$a_n = g \tan(\varphi + e) \approx g(\varphi + e) \tag{3.7.1}$$

a_c = [illegible] sin ϕ, tan([illegible]) = [illegible]



S_o

$$\frac{a_l}{g} = \sqrt{1 + \left(\frac{a_n}{g}\right)^2} \sin\left(\tan^{-1}\frac{a_n}{g} - e\right)$$

for $e = 12°$

if $\frac{a_l}{g} = .125$, $\frac{a_n}{g} = .3403$

$= .250$ $\quad = .4681$

Figure 3-5. A Superelevated Guideway

in which the small angle approximation is sufficiently accurate.

From equation (3.1.13), the minimum radius of curvature is given by

$$R = \frac{V^2}{a_n} \quad = \frac{V^2}{.3403\,g}, \; \frac{V^2}{.4681\,g}$$

standing seated

if a_n is the largest permissible value of this quantity. Substituting from equation (3.7.1),

$$R_{\min} = \frac{V^2}{g(\varphi \pm e)} \tag{3.7.2}$$

tan

in which φ is the maximum permissible lateral acceleration in a curve divided by g.

The permissible angle e is limited by the possibility that the vehicle might have to stop on curves to a value of about 12 degrees or 0.2 radian. With standing passengers, φ is limited to about 1/8 radian, and if all passengers are seated to about 1/4 radian. Thus, with superelevation the minimum radius of curvature can be reduced by the ratio

$$\frac{R_{e=0}}{R_{e\neq 0}} = 1 + \frac{e}{\varphi} \quad \begin{matrix} = 2.72 \text{ standing} \\ = 1.87 \text{ seated} \end{matrix} \tag{3.7.3}$$

For standing passengers and $e = 0.2$ radian, $e/\varphi = 1.6$, and for seated passengers, $e/\varphi = 0.8$. Thus, the reduction in $R_{\min}$ is very significant and worth pursuing. In designing a superelevated curve, the spiral (or varying curvature) transition section must be twisted as well as curved in the horizontal plane. The angle of twist is zero at the zero-curvature end of the spiral and increases uniformly to a value of e of about 12 degrees at the end of maximum curvature.

for fixed R, $V \sim \sqrt{2.72} = 1.65$ standing

$\sqrt{1.87} = 1.37$ seated

3.8 Summary

In the layout design of almost every guideway transit system, some sections of curved guideway are necessary. The design of specific systems must therefore be delayed until the student has an appreciation of the design of transitions from straight to curved guideway. These transitions must be designed so that the magnitude of lateral motions are acceptable from the standpoint of comfort. Comfort depends on keeping the maximum lateral acceleration and rate of change of acceleration (jerk) below specified values. This results in the requirement that transitions from straight guideways to guideways of constant curvature must be separated by sections of constantly increasing curvature or spirals. If the speed is constant throughout the transition, the spiral section can generally be approximated by a simple cubic given by equation (3.2.11). Two important applications of the cubic transition are derived: (1) the right-angle curve, and (2) the transition to a parallel guideway, such as used in entry into an off-line station. In both cases enough information is given so that each of these types of curves can be specified. From the equations derived, it is straightforward to derive the transition between two straight lines of arbitrary angle.

Curved guideway costs more than straight guideway, therefore it is desirable to reduce the length of curved guideway wherever possible. In the transition to an off-line station, this is possible if the vehicle starts to decelerate before entering the transition, because sharper curves can be negotiated at the same level of comfort at lower speeds. Thus, if the transition curve is designed to take advantage of the lower speed, it will be shorter. The solution to this problem is lengthy, but it is included because of its importance in reducing guideway cost in certain applications. Instead of a simple cubic, the transition curve is given by the more complex expression, equation (3.5.19).

Spiral guideway can be more expensive to manufacture than guideway of constant curvature, therefore it is useful to know under what circumstances it is possible to approximate a spiral section by one or more sections of constant curvature. Such a transition may be possible within the jerk-comfort limit if the lateral suspension system of the vehicles can compensate for the lack of a spiral transition. Thus the problem is solved by considering a vehicle with given lateral suspension dynamics negotiating an abrupt change in curvature in the guideway. For the case of a linear spring-dashpot suspension system, the solution is worked out in detail. Equation (3.6.17) and figure 3-4 show that the greatest change in curvature can be permitted if the damping ratio if the lateral suspension system is one third. With this damping ratio, the minimum tolerable radius of curvature is given as a function of line speed, maximum tolerable jerk, and maximum suspension system deflection by equation (3.6.20). This equation possesses an interpretation of more general interest: It shows that, with a given change in curvature and a given suspension system, the maximum jerk experienced by the passengers is proportional to the cube of the speed. If a change in curvature is considered as a typical imperfection in the straightness of the guideway due to manufacturing tolerances, erection tolerances, or ground shifts, then in general the discomfort of the ride in terms of jerk worsens in proportion to the cube of the speed, and indicates why it is so much more important to keep the track straight at high speeds. The required tolerances are relaxed if the lateral deflection capability of the lateral suspension system is as large as possible, and if the damping ratio is properly chosen. More analysis is needed to determine if the optimum value of one third computed for the case considered would be different with different kinds of imperfections.

Finally, superelevation as a method of reducing the length and radius of curves is considered in enough detail to provide necessary design information. It is shown that the reduction in the radius of curvature practically possible is a factor of about 1.8 for seated passengers, and 2.6 for standing passengers. Thus, superelevation is well worth considering.

Problems

1. A seated-passenger guideway vehicle system is to be designed to permit right-angle turns at constant speed on city streets for which clearance available for the guideways is 40 m to the centerline of the guideways, that is, if two sets of parallel lines 40 m apart are drawn perpendicular to each other, the centerline of the guideway in making a right-angle turn must lie inside the boundaries of these lines.

   a. Sketch the curve and label all parts.
   b. With no superelevation, what is the maximum velocity for which the curve can be designed.
   c. Assuming the maximum velocity, compute the coordinates of the endpoints of the transition segments with respect to the street corner intersected 45° through the curve, and plot the curve.
   d. If the normal line speed is 20 m/s, what is the deceleration length that must be allowed for before the curve is negotiated. explain
   e. What is the length of each spiral section, and what is the length of the section of constant curvature? (Make all length computations to the nearest cm.)

2. For a seated-passenger vehicle system in which the line speed is 20 m/s, design a constant velocity transition to a parallel guideway ten meters away.

a. Make a careful sketch of the transition and label all parts. (Note comment following equation (3.4.5).)
b. Compute the coordinates and slopes of all points between transition sections and lay out the transition curve on graph paper making use of symmetry properties where possible.
c. Write an equation for the total length of the transition in terms of V, H, and a_n, and compute the length. By what percentage is the length greater than the maximum length of an all-spiral transition at the same velocity?
d. What is the percent error in length of the transition as computed by equation (3.2.10) instead of by the exact formula (3.2.9). (Hint: integrate the second term in the expansion of $\cos\theta$.)

3. With standing-passenger vehicles, a constant deceleration spiral is designed to turn the guideway through the maximum possible angle with $V_0 = 10$ m/s.

 a. Compute V_{min}.
 b. Compute θ_{max}.
 c. Compute the length of the spiral section.
 d. Compute the $x - y$ coordinates of the end point.
 e. What is the length of a constant velocity spiral with $V = V_0$ and the same θ_{max}?

4. For standing-passenger vehicles with lateral suspension systems having a maximum permissible deflection of 10 cm and an optimum damping ratio, the line speed is 15 m/s. It is desired to build a right-angle turn using a minimum length of curved guideway, using segments of two different constant curvatures, and maintaining constant line speed.

 a. Sketch the curve and label all parts.
 b. If two cycles of oscillation of the lateral suspension system must be completed before entering the second segment, what is the length of each segment.
 c. What is the total length of the minimum length right-angle curve?
 d. What are the coordinates of the endpoint of the total right-angle transition with respect to the initial point? (Hint: Follow the derivations of equations (3.3.2) to (3.3.7.)

5. Assume the constant curvature section of the transition curve of Problem 1 is superelevated a maximum permissible amount. With the same velocity as computed in Problem 1, how much narrower could the width of the streets have been as a percentage of the given width? How much higher could the velocity have been with the same width of streets as a percentage of the velocity with no superelevation?