

What is PRT?

J. Edward Anderson, PhD, P. E.

Based on discussions at the September 20, 2019 ATRA Board Meeting about the ASCE Standard's Committee decision to drop the Brick-Wall-Stop Criterion in favor of requiring the development company to demonstrate safety, I thought I would provide some background that perhaps only one who has been involved with PRT for over 50 years would know. I got involved in PRT in 1968 for reasons given in the accompanying paper "The Origin of My Interest in PRT." In 2009 Ingmar asked me to prepare a paper entitled "Overcoming Headway Limitations in Personal Rapid Transit Systems" to give at the 2009 Pod Car Conference in Malmo, Sweden. It can be found on www.advancedtransit.org/Library/Papers and is key to this discussion.

Now, at the age of 92, I look back to the men who were my principal motive for continuing to work on PRT:

Jerry Kieffer, who I met in 1969 and who in the 1980s and 90s was the glue that kept ATRA going. During the 1970s he served in a high position in the Agency for International Development and later as Deputy Administrator of the Social Security Administration. He unfortunately for all of us contracted Parkinson's and died of it a few years ago in his 90s.

Jack Irving, Vice President of The Aerospace Corporation, who began working on PRT in 1968 with a strong team of engineers. Without his work, I would have stopped working on PRT long ago. He died at the age of 88 from diabetes.

Larry Goldmuntz, whom I met while he was Director of Civilian Technology in the Executive Office of the President. He suggested at the 3rd International PRT Conference in 1975 that the conference committee organize an AGT Society, which became ATRA. He served for a time as ATRA Chairman. He died of a heart attack at the age of 75 while on vacation in Italy.

Mike Powells, the first Chairman of ATRA. He was Vice President of Barton Aschman Associates. In that position, he was able to make progress on behalf of ATRA with UMTA Associate Administrator George Pastor. Shortly after he retired, he died in a fall down the stairway in his home in Evanston, Illinois.

Tom Floyd, at a critical time Chairman of ATRA. As a former associate administrator of UMTA, he played a key role in interesting the Chairman of the Northeastern Illinois Regional Transportation Authority in investing in PRT. He and his wife died in an auto accident while starting a vacation in Norway.

Byron Johnson, for a time both President and then Chairman of ATRA. In his other lives he had been a Colorado Congressman, Chairman of the Board of Regents of the University of Colorado and later Chairman of the Colorado Regional Transportation Board. On many visits to Denver, he and I worked together promoting PRT in many ways. He died at the age of 80, of I am quite sure a broken heart after his wife died a year before.

I apologize for not mentioning others who I should have remembered. I keep going without yet any of the afflictions that took the above friends but knowing that I will be approaching in the not too distant future my own twilight zone. Fortunately, I have a 36-year-old grandson who has been studying my work on PRT for many years. He has engaged a large engineering company that will fund 80% of ITNS projects. Incidentally, Taxi 2000 Corporation ceased their operations on June 30, 2017. If you would like to see the notice let me know.

My Task Force on New Concepts in Urban Transportation in 1970 very quickly caught the PRT bug and, as concluded in the paper “How to Reduce Congestion,”¹ in addition to the usual definition of PRT, thinking of area-wide systems we found it necessary to be able to operate safely at half-second headways. As shown on page 16 of the attached paper “The Origin of My Interest in PRT,” in 1973 UMTA agreed with that requirement. In time it became clear that such a short headway could not be achieved safely in all weather conditions in a system in which the vehicles accelerate and brake in a way dependent on the friction at the running surface. In 1968, when Jack Irving started his PRT program, linear induction motors, which do not depend on friction at the running surface, had not developed sufficiently, hence he and his group developed a linear pulsed d. c. motor.² The key advance came in the late 1970s – the variable frequency drive. With it, the efficiency of the linear induction motor (LIM) is markedly increased. A paper on LIMs, “Properties of a Linear Induction Motor,” is included herein on pages 5-14.

I attach the document “Aerospace Study of ATN in San Jose,” which is the 2012 report “Automatic Transit Network Feasibility for the San Jose airport.” On page 75 they give the minimum headway for then operational ATN systems as 6 seconds. Because 6 seconds minimum headway was much too long to meet San Jose Airport needs, Aerospace recommended none of the then operational “PRT” systems. Now, 7 years later, San Jose has not installed any ATN system. They are waiting for a system capable of much shorter minimum headway, and we know now that that requires use of linear induction motors for acceleration and braking. Thus, a true definition of real PRT must include specification of a much shorter minimum headway. With LIMs, half-second minimum headway is fully practical as the UMTA team determined in 1973 – 47 years ago the needed headway was considered both required and safe. Wow! Technology really advances slowly.

¹ www.advancedtransit.org/Library Pages 17 and 18.

² Irving, J. H., Bernstein, H., Olson, C. L., and Buyan, J. *Fundamentals of Personal Rapid Transit*, Lexington Books, D. C. Heath and Company, Lexington, MA, 1978. www.advancedtransit.org/Library

The Linear Induction Motor

The basic principles of electromagnetic induction upon which a linear induction motor operates are illustrated in Figure 1. If a magnet moves at a speed V the magnetic flux ahead of the magnet increases in time and the magnetic flux behind the magnet decreases with time. The way Mother Nature works, a time varying magnetic field induces an electric field proportional to the rate of change of the magnetic field. If the magnet moves along a conducting surface, the induced electric field causes a current in the plate proportional to it. With the North and South poles as shown in Figure 1, the current is counterclockwise ahead of the moving magnet and clockwise behind. Under the middle of the magnet the induced currents come out of the paper towards the left in Figure 1 and here the magnetic field is of maximum strength and is downward. This is the principle of electromagnetic induction, which was discovered by Michael Faraday in the 1830s.

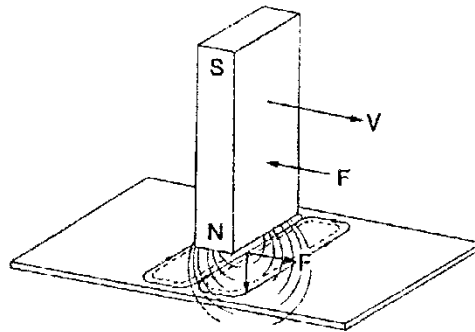


Figure 1. Current induced by a moving magnet.

A property of nature is that when an electric current passes across a magnetic field a force is produced perpendicular to both the current and the magnetic field. James Clerk Maxwell, in the mid-1850s, derived the laws under which these phenomena work in a set of equations now called Maxwell's equations. The remarkable fact is that these equations have proved time and again to be exact. In Figure 1 the force on the plate is to the right. If the conducting plate is resting on a low-friction surface, the plate will be dragged along with the magnet.

To achieve the speed V , the person or object moving the magnet must overcome the force F on the magnet, which force is equal and opposite to the force induced in the conducting plate. If the conducting plate is fastened down and above it there is a vehicle containing a traveling magnetic field moving to the right, the force F moves the vehicle to the left. It is thus necessary to devise a way to produce such a traveling magnetic field.

A means for producing a traveling magnetic field is illustrated in Figure 2. In the middle illustration, the elongated object containing slots is one of a stack of thin iron sheets stamped out with a die. The bottom view shows the same iron sheets in edge view stacked up as in a transformer. Three sets of coils are wound as indicated and are connected as shown in either the illustration on the left or on the right. The left illustration shows what is called a "delta" connection after the Greek letter Δ that it resembles, and the one on the right is a "Y" connection. A three-phase alternating voltage is applied to the three windings with a frequency that increases with

speed, with the voltages phased as shown in the top diagram. With the delta connection the voltage is applied directly across each set of coils. With the Y-connection each phase of the voltage is applied across two coils. The delta connection may therefore be used for high thrust and the Y connection for moderate thrust.

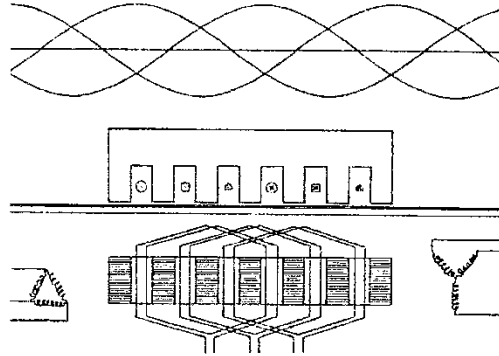


Figure 2. A means for producing a travelling magnetic field.

There is a frequency of oscillation of the voltage that minimizes the current and that frequency increases linearly with speed. See page 13. This is important because the electrical resistance losses are proportional to the square of the current, and thus the surface needed to dissipate the heat generated is proportional to the square of the current. The drive's volume increases as the three halves power of its surface area. Hence the required volume of the drive must increase in proportion to the cube of the maximum current. For this reason, maintaining the frequency of oscillation of the voltage close to optimum is very important. Linear induction motors have been under development for many decades and several books have been written describing them in technical detail.



This picture shows the chassis of the ITNS vehicle I designed.

Two basic ideas: 1) A vertical chassis, and 2) LIM propulsion and braking.

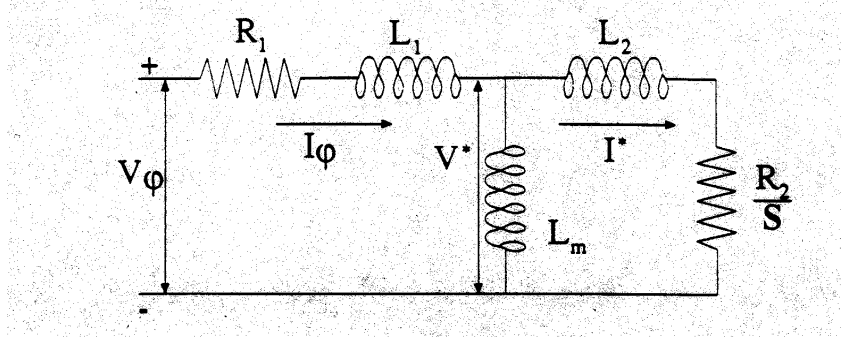
The air-cooled LIMs to be mounted under the chassis are lying on a wooden platform on the left.

The green boxes mounted on the chassis are variable frequency drives, each of which drives one LIM.

These LIMs were built by Force Engineering, Ltd in England. www.force.co.uk

Properties of a Linear Induction Motor

The purpose of this paper is to show how the relationships between voltage and current as well as other properties of a low-speed flat LIM can be found from the equivalent circuit model shown here.³:



The R's are resistances, the L's inductances, the I's currents, and the V's voltages. S stands for slip. The voltage-current relationships for this circuit are

$$V^* = L_m \frac{d(I_\phi - I^*)}{dt} = L_2 \frac{dI^*}{dt} + \frac{R_2}{s} I^* \quad (1)$$

$$V_\phi - V^* = R_1 I_\phi + L_1 \frac{dI_\phi}{dt} \quad (2)$$

in which

$$s = \text{slip} = \frac{U_s - U}{U_s} \quad (3)$$

U = speed, m/s

$$U_s = \text{synchronous speed, } 2\tau f, \text{ m/s} \quad (4)$$

τ = pole pitch, m

$$f = \frac{\omega}{2\pi} = \text{frequency, Hz} \quad (5)$$

From equation (1)

$$L_m \frac{dI_\phi}{dt} = (L_m + L_2) \frac{dI^*}{dt} + \frac{R_2}{s} I^* \quad (6)$$

From equation (2) and the first of equations (1)

³ Circuit diagram from the book "Linear Motion Electric Machines" by I. Boldea and S. A. Nasar.

$$V_\phi = R_1 I_\phi + (L_m + L_1) \frac{dI_\phi}{dt} - L_m \frac{dI^*}{dt} \quad (7)$$

Substitute dI_ϕ / dt from equation (6) into equation (7). Then

$$V_\phi = R_1 I_\phi + \left(L_1 + L_2 + \frac{L_1 L_2}{L_m} \right) \frac{dI^*}{dt} + \left(1 + \frac{L_1}{L_m} \right) \frac{R_2}{s} I^*$$

Differentiate and substitute equation (6). Then

$$\frac{dV_\phi}{dt} = \frac{R_1}{L_m} \left[(L_m + L_2) \frac{dI^*}{dt} + \frac{R_2}{s} I^* \right] + \left(L_1 + L_2 + \frac{L_1 L_2}{L_m} \right) \frac{d^2 I^*}{dt^2} + \left(1 + \frac{L_1}{L_m} \right) \frac{R_2}{s} \frac{dI^*}{dt}$$

which can be rewritten as

$$\left(L_1 + L_2 + \frac{L_1 L_2}{L_m} \right) \frac{d^2 I^*}{dt^2} + \left[R_1 \left(1 + \frac{L_2}{L_m} \right) + \frac{R_2}{s} \left(1 + \frac{L_1}{L_m} \right) \right] \frac{dI^*}{dt} + \frac{R_1 R_2}{s L_m} I^* = \frac{dV_\phi}{dt} \quad (8)$$

Equation (6) can be rewritten in the form

$$\frac{dI_\phi}{dt} = \left(1 + \frac{L_2}{L_m} \right) \frac{dI^*}{dt} + \frac{R_2}{s L_m} I^* \quad (9)$$

The Root-Mean-Square values of Current and Voltage

Let the r.m.s. amplitudes of the voltage and currents be expressed as $V_{\phi_{rms}}, I_{\phi_{rms}}, I_{rms}^*$. For the non-electrical engineer, note that if $V = \bar{V} \sin \omega t$, the r.m.s. value of voltage is defined as the square root of the average of the square of the voltage over a half cycle. Thus

$$V_{rms}^2 = \frac{\omega}{\pi} \int_0^{\pi/\omega} V^2 dt = \frac{\bar{V}^2}{\pi} \int_0^\pi \sin^2 \omega t d(\omega t) = \frac{\bar{V}^2}{2}$$

Therefore

$$V_{rms} = \frac{1}{\sqrt{2}} \bar{V}.$$

For example, the power dissipated in a circuit with current $I = \bar{I} \sin(\omega t - \phi)$ is

$$Power = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} IV dt = \frac{\bar{I}\bar{V}}{2\pi} \int_0^{2\pi} \sin \omega t \sin(\omega t - \phi) d(\omega t)$$

Let $\theta = \omega t$. Then

$$Power = \frac{\overline{IV}}{2\pi} \left(\cos \varphi \int_0^{2\pi} \sin^2 \theta d\theta - \sin \varphi \int \sin \theta \cos \theta d\theta \right) = \frac{\overline{IV}}{2} \cos \varphi$$

Substitute

$$V_{rms} = \frac{\overline{V}}{\sqrt{2}}, \quad I_{rms} = \frac{\overline{I}}{\sqrt{2}}$$

Hence

$$Power = I_{rms} V_{rms} \cos \varphi$$

where $\cos \varphi$ is the power factor. Thus, if we always work in terms of r.m.s. values, the factor $\sqrt{2}$ does not appear. In equations (8) and (9) the factor $\sqrt{2}$ would drop out in any case.

Solution of the Current/Voltage Equations

Now let

$$V_\phi = \sqrt{2} V_{\phi_{rms}} e^{i\omega t}, \quad I_\phi = \sqrt{2} I_{\phi_{rms}} e^{i\omega t}, \quad I^* = \sqrt{2} I_{rms}^* e^{i\omega t} \quad (10)$$

where the quantities $V_{\phi_{rms}}$, etc. are complex. Since the factors $\sqrt{2}e^{i\omega t}$ divide out of equations (8) and (9), there is no ambiguity in dropping the subscript rms. Thus, from equations (8) and (9) we find

$$I^* = \frac{i\omega V_\phi}{-C_1\omega^2 + i\omega C_2 + C_3} \quad (11)$$

$$I_\phi = C_4 I^* - \frac{i}{\omega} C_5 I^* \quad (12)$$

in which

$$\begin{aligned} C_1 &= L_1 + L_2 + \frac{L_1 L_2}{L_m} \\ C_2 &= R_1 \left(1 + \frac{L_2}{L_m} \right) + \frac{R_2}{s} \left(1 + \frac{L_1}{L_m} \right) \\ C_3 &= \frac{R_1 R_2}{s L_m} \\ C_4 &= 1 + \frac{L_2}{L_m} \\ C_5 &= \frac{R_2}{s L_m} \end{aligned} \quad (13)$$

These are temporary coefficients and will not be used in the results.

Write equation (11) in the form

$$I^* = \frac{i\omega V_\phi}{A + iB} \quad (14)$$

in which

$$A = C_3 - C_1\omega^2, \quad B = \omega C_2. \quad (15)$$

Then equation (14) can be written

$$I^* = \frac{i\omega V_\phi (A - iB)}{A^2 + B^2} = \frac{\omega V_\phi}{A^2 + B^2} (B + iA)$$

Substituting into equation (12)

$$I_\phi = \frac{\omega V_\phi}{A^2 + B^2} (B + iA) \left(C_4 - i \frac{C_5}{\omega} \right)$$

or

$$I_\phi = \frac{\omega V_\phi}{A^2 + B^2} \left[BC_4 + \frac{AC_5}{\omega} + i \left(AC_4 - \frac{BC_5}{\omega} \right) \right] \quad (16)$$

Then the square of the magnitude of the current is

$$\begin{aligned} |I_\phi|^2 &= \frac{\omega^2 V_\phi^2}{(A^2 + B^2)^2} \left(B^2 C_4^2 + 2AB \frac{C_4 C_5}{\omega} + A^2 \frac{C_5^2}{\omega^2} + A^2 C_4^2 - 2AB \frac{C_4 C_5}{\omega} + B^2 \frac{C_5^2}{\omega^2} \right) \\ &= \frac{\omega^2 V_\phi^2}{A^2 + B^2} \left(C_4^2 + \frac{C_5^2}{\omega^2} \right) \end{aligned} \quad (17)$$

Since input power is the voltage multiplied by the real part of the current, note that, after substituting the values of A and B from equation (15), the real component in equation (16) becomes

$$BC_4 + A \frac{C_5}{\omega} = \omega C_2 C_4 + (C_3 - C_1 \omega^2) \frac{C_5}{\omega} = \frac{1}{\omega} \left[\omega^2 (C_2 C_4 - C_1 C_5) + C_3 C_5 \right]$$

in which

$$C_2 C_4 - C_1 C_5 = R_1 \left(1 + \frac{L_2}{L_m} \right)^2 + \frac{R_2}{s}$$

So

$$BC_4 + A \frac{C_5}{\omega} = \frac{1}{\omega s^2} \left\{ \omega^2 s \left[sR_1 \left(1 + \frac{L_2}{L_m} \right)^2 + R_2 \right] + R_1 \left(\frac{R_2}{L_m} \right)^2 \right\} \quad (18)$$

Now

$$A^2 + B^2 = C_1^2 \omega^4 + \omega^2 (C_2^2 - 2C_1 C_3) + C_3^2$$

$$C_2^2 - 2C_1 C_3 = R_1^2 \left(1 + \frac{L_2}{L_m} \right)^2 + \frac{R_2^2}{s^2} \left(1 + \frac{L_1}{L_m} \right)^2 + \frac{2R_1 R_2}{s}$$

so

$$A^2 + B^2 = \frac{\omega^2}{s^2} \left[\left(L_1 + L_2 + \frac{L_1 L_2}{L_m} \right)^2 \omega^2 s^2 + R_1^2 \left(1 + \frac{L_2}{L_m} \right)^2 s^2 + R_2^2 \left(1 + \frac{L_1}{L_m} \right)^2 + 2R_1 R_2 s + \frac{1}{\omega^2} \left(\frac{R_1 R_2}{L_m} \right)^2 \right] \quad (19)$$

Define a new set of coefficients:

$$D_1 = \left(L_1 + L_2 + \frac{L_1 L_2}{L_m} \right)^2$$

$$D_2 = R_1^2 \left(1 + \frac{L_2}{L_m} \right)^2$$

$$D_3 = R_2^2 \left(1 + \frac{L_1}{L_m} \right)^2$$

$$D_4 = 2R_1 R_2$$

$$D_5 = \left(\frac{R_1 R_2}{L_m} \right)^2$$
(20)

Let

$$P = D_1 \omega^2 s^2 + D_2 s^2 + D_3 + D_4 s + \frac{1}{\omega^2} D_5 \quad (21)$$

$$Q = D_2 s^2 + \frac{1}{\omega^2} D_5 \quad (22)$$

Then

$$A^2 + B^2 = \frac{\omega^2}{s^2} P \quad (23)$$

Substituting $A^2 + B^2$ and the definitions of C_4 and C_5 from equations (13) into equation (17), the equation for the magnitude of phase current becomes

$$\begin{aligned}
|I_\phi| &= \frac{\omega V_\phi}{R_1 \sqrt{A^2 + B^2}} \sqrt{R_1^2 \left(1 + \frac{L_2}{L_m}\right)^2 + \frac{R_1^2}{\omega^2} \left(\frac{R_2}{sL_m}\right)^2} \\
&= \frac{s\omega V_\phi}{R_1 \omega \sqrt{P}} \sqrt{D_2 + \frac{D_5}{\omega^2 s^2}} = \frac{V_\phi}{R_1 \sqrt{P}} \sqrt{D_2 s^2 + \frac{D_5}{\omega^2}}
\end{aligned}$$

Substituting from equation (22)

$$|I_\phi| = \frac{V_\phi}{R_1} \sqrt{\frac{Q}{P}} \quad (24)$$

Now, from equation (16) and substituting equations (18) and (23), the real portion of the current is

$$\begin{aligned}
\text{Re}(I_\phi) &= \frac{\omega V_\phi s^2}{\omega^2 P} \frac{1}{\omega s^2} \left\{ \omega^2 s \left[sR_1 \left(1 + \frac{L_2}{L_m}\right)^2 + R_2 \right] + R_1 \left(\frac{R_2}{L_m}\right)^2 \right\} \\
&= \frac{V_\phi}{R_1 P} \left[s^2 D_2 + \frac{1}{2} s(2R_1 R_2) + \frac{1}{\omega^2} \left(\frac{R_1 R_2}{L_m}\right)^2 \right] \\
&= \frac{V_\phi}{R_1 P} \left(s^2 D_2 + \frac{D_5}{\omega^2} + \frac{1}{2} s D_4 \right) = \frac{V_\phi}{R_1 P} \left(Q + \frac{1}{2} s D_4 \right)
\end{aligned}$$

so

$$\text{Re}(I_\phi) = \frac{V_\phi}{R_1} \left(\frac{Q + sD_4/2}{P} \right) \quad (25)$$

$$\text{Primary Power Loss} = R_1 |I_\phi|^2 = \frac{V_\phi^2}{R_1} \frac{Q}{P} \quad (26)$$

$$\text{Input Power} = V_\phi \text{Re}(I_\phi) = \frac{V_\phi^2}{R_1} \left(\frac{Q + sD_4/2}{P} \right) = \frac{V_\phi^2}{R_1} \left(\frac{Q + sR_1 R_2}{P} \right) \quad (27)$$

$$\text{Power Factor} = \frac{\text{Input Power}}{\text{Input Volt} - \text{Amps}} = \frac{\frac{V_\phi^2}{R_1} \left(\frac{Q + sD_4/2}{P} \right)}{\frac{V_\phi}{R_1} \left(\frac{Q}{P} \right)^{1/2}} = \frac{Q^{1/2} + 0.5sD_4 Q^{-1/2}}{P^{1/2}} \quad (28)$$

To determine the Secondary Loss, consider the equation for conservation of energy:

Input Power = Primary Loss + Secondary Loss + Thrust × Speed

After substituting from equations (27), (26) and (3) we get

$$\frac{V_\phi^2}{R_1} \left(\frac{Q + sR_1R_2}{P} \right) = \frac{V_\phi^2}{R_1} \frac{Q}{P} + \text{Secondary Loss} + TU_s(1-s)$$

or
$$V_\phi^2 \frac{R_2}{P} s = SL + TU_s(1-s) \quad (29)$$

in which T is thrust, Speed = $U_s(1-s)$, and SL = Secondary Loss. Note that when

$s = 1$, $SL = V_\phi^2 \frac{R_2}{P}$; $s = 0$, $SL = T = 0$. To meet these conditions and equation (29) we must have

$$\text{Secondary Loss} = V_\phi^2 \frac{R_2}{P} s^2 \quad (30)$$

Substituting equation (30) into equation (29), the thrust is

$$\text{Thrust} = \frac{R_2 V_\phi^2}{U_s P} s = \frac{\pi}{\tau \omega} \frac{R_2 V_\phi^2}{P} s \quad (31)$$

in which, from equations (4) and (5), we find that $U_s = \tau \omega / \pi$. Note that these formulae apply to ONE PHASE.

Finally,

$$\text{Efficiency} = \frac{\text{Developed Power}}{\text{Input Power}} = \frac{\frac{R_2 V_\phi^2}{P} s(1-s)}{\frac{V_\phi^2}{R_1} \left(\frac{Q + R_1 R_2}{P} \right)} = \frac{R_1 R_2 s(1-s)}{Q + R_1 R_2 s} \quad (32)$$

SUMMARY

$$D_1 = \left(L_1 + L_2 + \frac{L_1 L_2}{L_m} \right)^2$$

$$D_2 = R_1^2 \left(1 + \frac{L_2}{L_m} \right)^2$$

$$D_3 = R_2^2 \left(1 + \frac{L_1}{L_m} \right)^2$$

$$D_4 = 2R_1 R_2$$

$$D_5 = \left(\frac{R_1 R_2}{L_m} \right)^2$$

$$P = D_1\omega^2s^2 + D_2s^2 + D_3 + D_4s + \frac{1}{\omega^2}D_5$$

$$Q = D_2s^2 + \frac{1}{\omega^2}D_5$$

$$\text{Phase Current} = \frac{V_\phi}{R_1} \sqrt{\frac{Q}{P}}$$

$$\text{Input Power per Phase} = \frac{V_\phi^2}{P} \left(\frac{Q}{R_1} + sR_2 \right)$$

$$\text{Developed Power per Phase} = \frac{R_2V_\phi^2}{P} s(1-s)$$

$$\text{Primary Loss per Phase} = \frac{V_\phi^2}{R_1} \frac{Q}{P}$$

$$\text{Secondary Loss per Phase} = V_\phi^2 \frac{R_2}{P} s^2$$

$$\text{Thrust per Phase} = \frac{\pi}{\tau\omega} \frac{R_2V_\phi^2}{P} s$$

$$\text{Power Factor} = \left(Q^{1/2} + R_1R_2sQ^{-1/2} \right) P^{-1/2}$$

$$\text{Efficiency} = \frac{s(1-s)}{s + Q/R_1R_2}$$

Current Minimization

Equation (31) solved for the phase voltage is

$$V_\phi = \left(\frac{T_\phi \tau \omega P}{\pi R_2 s} \right)^{1/2} \quad (33)$$

Substitute equation (33) into equation (24)

$$|I_\phi| = \frac{V_\phi}{R_1} \sqrt{\frac{Q}{P}} = \frac{1}{R_1} \left(\frac{T_\phi \tau \omega Q}{\pi R_2 s} \right)^{1/2} = \frac{1}{R_1} \left(\frac{T_\phi \tau}{\pi R_2} \right)^{1/2} \left(\frac{\omega Q}{s} \right)^{1/2}$$

Substitute for Q from equation (22). Then the magnitude of the phase current becomes

$$|I_\phi| = \frac{1}{R_1} \left(\frac{T_\phi \tau}{\pi R_2} \right)^{1/2} \left(D_2 \omega s + \frac{D_5}{\omega s} \right)^{1/2} \quad (34)$$

Note that there is a value of ωs that minimizes the phase current and it is found by differentiating the expression on the right of equation (34). Thus

$$\frac{\partial}{\partial \omega s} \left(D_2 \omega s + \frac{D_5}{\omega s} \right) = D_2 - \frac{D_5}{(\omega s)^2}$$

Setting the expression on the right to zero yields the value of $\omega s \equiv (\omega s)_{opt}$ that minimizes the phase current. Thus

$$(\omega s)_{opt} = \sqrt{\frac{D_5}{D_2}} = \frac{R_2}{L_m + L_2} \quad (35)$$

in which values for D_5 and D_2 have been taken from equations (20).

How do we know that equation (35) gives a minimum not a maximum? Because

$$\frac{\partial}{\partial \omega s} \left(D_2 - \frac{D_5}{(\omega s)^2} \right) = \frac{2D_5}{(\omega s)^3} > 0.$$

From equations (3), (4), and (5)

$$\omega s = \omega - \frac{\pi}{\tau} U \quad (36)$$

Substituting equation (36) into equation (35), we see that the optimum frequency, i.e., the frequency that minimizes current is

$$\omega_{opt} = \frac{R_2}{L_m + L_2} + \frac{\pi}{\tau} U$$

Using equation (5), the optimum frequency is

$$f_{opt} = f_o + \frac{U}{2\tau}, \quad \text{where} \quad f_o = \frac{R_2}{2\pi(L_m + L_2)} \quad (37)$$

If we substitute $(\omega s)_{opt}$ from equation (35) and the definitions of D_5 and D_2 into equation (34) the result for the minimum current for a given thrust can be written in the form

$$|I_\phi|_{\min} = \left[\frac{2 T_\phi \tau}{\pi L_m} \left(1 + \frac{L_2}{L_m} \right) \right]^{1/2} \quad (38)$$

and we note that the minimum current depends only on thrust, not speed. Solving equation (38) for thrust and taking into account three phases, we see that for any limit phase current, the maximum thrust that can be obtained with given values of τ , L_m , L_2 is

$$T_{\max} = \frac{3\pi}{2} \frac{L_m}{\tau} \left(\frac{L_m}{L_m + L_2} \right) I_{\phi_{\text{lim}}}^2. \quad (39)$$

I derived these equations in August 1984 a few weeks before I flew to England to visit two companies that built LIMs. Both companies were the result of the work of Professor Eric Laithwaite. You can read about him on Wikipedia. One of these companies, force.co.uk, gave me a printout of the properties of their LIMs. They used the same notation I did. When I got home, I took their values for the inductances and resistances shown in the figure on page 5 and plugged them into my equations. The results agreed to four decimal places!

